133. A Characterization of L^2-well Posedness for Iterations of Hyperbolic Mixed Problems of Second Order

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(Communicated by Kôsaku YOSIDA, M. J. A., Nov. 12, 1976)

§1. **Introduction and theorem.** We are concerned with an iterated mixed problem as follows:

$$(\tilde{P}, \tilde{B}_1, \cdots, \tilde{B}_m)\begin{cases}\tilde{P}(x, D)u=f & \text{in } \Omega,\\ \tilde{B}_j(x', D)u=g_j & \text{on } \Gamma, j=1, \cdots, m.\end{cases}$$

Here Ω and Γ are the open half space $\{x=(x', x_n)=(x_0, x'', x_n)\,;\, x_0 \in R^1, x'' \in R^{n-1}, x_n>0\}$ $(n\geqq 2)$ and its boundary respectively, and for covariable (τ, σ, λ) of (x_0, x'', x_n) the principal symbols $\tilde{P}^0(x, \tau, \sigma, \lambda)$, $\tilde{B}^0_j(x', \tau, \sigma, \lambda)$ of $\tilde{P}, \tilde{B}_j$ have the following forms:

$$\tilde{P}^0=P^0_1\cdots P^0_m,\ \tilde{B}^0_1=B^0_1,\ \tilde{B}^0_2=B^0_2P^0_1,\ \tilde{B}^0_3=B^0_3P^0_2P^0_1,\ \cdots,\ \tilde{B}^0_m=B^0_mP^0_{m-1}\cdots P^0_1,$$

where $P^0_j, j=1, \cdots, m$ are x_0-hyperbolic homogeneous operators of second order whose normal cones cut by $\tau=1$ don't intersect each other and are bounded surfaces in the (σ, λ) space for every fixed $x \in \Gamma$. Furthermore B^0_j is a homogeneous boundary differential operator at most of first order such that Γ is noncharacteristic for B^0_j. All the coefficients are assumed to be real and smooth in $\bar{\Omega}$ and to be constant near the infinity (see [2], [3], [8]).

Definition. *The problem $(\tilde{P}, \tilde{B}_1, \cdots, \tilde{B}_m)$ is said to be L^2-well posed if and only if there exist positive constants C and γ_0 such that for every $\gamma\geqq\gamma_0$ and $f \in H_{1,\gamma}(\Omega)$ the problem $(\tilde{P}, \tilde{B}_1, \cdots, \tilde{B}_m)$ with $g_j=0, j=1, \cdots, m$ has a unique solution u in $H_{2m,\gamma}(\Omega)$ satisfying*

$$\gamma \|u\|_{2m-1,\gamma} \leqq C \|f\|_{0,\gamma}. \tag{1.1}$$

(For function spaces see, e.g., [7]).

Now we have

Theorem. *The problem $(\tilde{P}, \tilde{B}_1, \cdots, \tilde{B}_m)$ is L^2-well posed if and only if all the frozen constant coefficients problems $(\tilde{P}^0, \tilde{B}^0_1, \cdots, \tilde{B}^0_m)_{x'}$ at boundary points $x' \in \Gamma$ are "uniformly L^2-well posed", that is, $(\tilde{P}^0, \tilde{B}^0_1, \cdots, \tilde{B}^0_m)_{x'}$ is L^2-well posed for every $x' \in \Gamma$ and the constants C in* (1.1) *with respect to these problems are independent of the parameter x'.*

§2. **Outline of the proof.** It is enough to prove the "if" part, because of Theorem 1 and Lemma 2.2 in [1]. Let $\tilde{L}(x', \tau, \sigma)$ and $L_j(x', \tau, \sigma), j=1, \cdots, m$ be the Lopatinskii determinants of $(\tilde{P}^0, \tilde{B}^0_1, \cdots, \tilde{B}^0_m)$ and (P^0_j, B^0_j) respectively. Then it follows from (3.2) and Theorem 1 in [2] respectively that

$$\tilde{L}=L_1\cdots L_m\cdot(\text{nonzero factor}) \tag{2.1}$$

and that every constant coefficients problem $(P_j^0, B_j^0)_{x'}(x'\in\Gamma, j=1,\cdots, m)$ is L^2-well posed. Hence we find by virtue of Lemma 4.1 in [2] that $\tilde{L}(x',\tau,\sigma)$ vanishes at a point $(x^0,\tau^0,\sigma^0)\in\Gamma\times(R^n\backslash 0)$ if and only if there is an index j such that $L_j(x^0,\tau^0,\sigma^0)=0$ and $P_j^0(x^0,\tau^0,\lambda)$ has a double real zero λ. Furthermore by our hypothesis with respect to the normal cones we see that such j is uniquely determind for given (x^0,τ^0,σ^0). Now we shall reduce $(\tilde{P},\tilde{B}_1,\cdots,\tilde{B}_m)$ to a system of first order which involves tangential pseudo-differential operators, by a usual transformation (see for instance [3]). Then the uniform L^2-well posedness of the problems $(\tilde{P}^0,\tilde{B}_1^0,\cdots,\tilde{B}_m^0)_{x'}, x'\in\Gamma$ implies the following inequalities:

$$\begin{aligned}(2.2)\quad &|b_{ij}(x',\tau,\sigma)|\\ &\leqq C\,|\mathrm{Im}\,\tau|^{-1}\,|\mathrm{Im}\,\lambda_i^+(x',\tau,\sigma)\,\mathrm{Im}\,\lambda_j^-(x',\tau,\sigma)|^{\frac{1}{2}}, i,j=1,\cdots,l,\end{aligned}$$

with a constant $C=C(x^0,\tau^0,\sigma^0)$ independent of not only (τ,σ) but x', where $\mathrm{Im}\,\tau<0$ and (x',τ,σ) varies near such a point $(x^0,\tau^0,\sigma^0)\in\Gamma\times(R^n\backslash 0)$ as $\tilde{L}(x^0,\tau^0,\sigma^0)=0$ (see Theorem 4.1, α), (i) in [9]). Here b_{ij} is the function defined in Definition 4.2 of [9] and $\lambda_j^\mp(x',\tau,\sigma), j=1,\cdots,m$ are zeros of $\tilde{P}^0(x',\tau,\sigma,\lambda)$ with positive imaginary part and negative one when $\mathrm{Im}\,\tau<0$ respectively such that $\lambda_j^\mp(x^0,\tau^0,\sigma^0), j=1,\cdots,m$ are simple real, double real or nonreal if $j<l, j=l$ or $j>l$ respectively. Furthermore it follows from (2.1) and [10], Lemma 2.2 that $\tilde{L}$ is decomposed as follows:

$$(2.3)\qquad \tilde{L}(x',\tau,\sigma)=(\sqrt{\tau-\theta(x',\sigma)}-D(x',\sigma))\tilde{L}^{(1)}(x',\tau,\sigma)$$

if $\mathrm{Im}\,\tau\leqq 0$, where $D(x^0,\sigma^0)=0$, $\tilde{L}^{(1)}(x^0,\tau^0,\sigma^0)\neq 0$, $\sqrt{\ }$ stands for the branch with $\sqrt{1}=1$ and θ is the real-valued function with $\theta(x^0,\sigma^0)=\tau^0$ defined in Lemma 3.1 of [9].

Now from (2.2) and (2.3) we find that

$$(2.4)\qquad \sum_{j=1}^{l-1}(|(b_{jl}\tilde{L})(x',\tau,\sigma)|^2+|(b_{lj}\tilde{L})(x',\tau,\sigma)|^2)\leqq C\,|D(x',\sigma)|,\qquad \text{if } \tau=\theta(x',\sigma),$$

where $C=C(x^0,\tau^0,\sigma^0)$. Furthermore by means of (2.20) and (2.21) in [10] we observe that (2.4) gives

$$(2.5)\qquad \sum_{j=1}^{l-1}(|(b_{jl}\tilde{L})(x',\tau,\sigma)|^2+|(b_{lj}\tilde{L})(x',\tau,\sigma)|^2\leqq -CQ(x',\tau,\sigma),\qquad \text{if } \tau=\theta(x',\sigma),$$

since $Q(x',\tau,\sigma)\leqq 0$ for such τ by the realness of the coefficients and L^2-well posedness of $(\tilde{P}^0,\tilde{B}_1^0,\cdots,\tilde{B}_m^0)_{x'}$. Here $C=C(x^0,\tau^0,\sigma^0)$ is a positive constant, Q is the function defined in Lemma 6.1 of [9] and we restrict ourselves to the case (a) in Lemma 3.1 of [9]. Notice that (2.5) is equivalent to (6.5) in [9], because of Definitions 4.1, 4.2 and (6.3) in [9]. Thus we can obtain a priori estimate for $(\tilde{P},\tilde{B}_1,\cdots,\tilde{B}_m)$, as in [9] or [3] (see (2.32) in [3]). The same argument may be applied to an adjoint problem and therefore we complete our proof (see Remark 1) in § 5 of [7]).

§ 3. **Remark.** In particular let $m=2$,

$$P_j^0(\tau, \sigma, \lambda)=\tau^2-a_j^2(|\sigma|^2+\lambda^2), \qquad j=1,2 \text{ and}$$
$$B_j^0(x', \tau, \lambda)=\lambda-C_j(x')\tau, \qquad j=1,2,$$

where a_j, $j=1,2$ are constants such that $0<a_2<a_1$ (see [2]). Then we have

Corollary. *The problem $(\tilde{P}, \tilde{B}_1, \tilde{B}_2)$ is L^2-well posed if and only if*

(3.1) $$C_j(x')\geqq 0, \qquad x'\in\Gamma, \qquad j=1,2,$$

and for every $x^0\in\Gamma$ with $C_1(x^0)=0$ there is a positive constant K such that

(3.2) $$C_2(x')^2\leqq KC_1(x') \qquad \textit{for } x' \textit{ near } x^0.$$

To prove the above fact it is enough to show that (3.2) is equivalent to (2.5) in this case. Let $(x^0, \tau^0, \sigma^0)\in\Gamma\times(R^n\backslash 0)$ be a point such that $L_1(x^0, \tau^0, \sigma^0)=0$ and $P_1^0(\tau^0, \sigma^0, \lambda)$ has a double real zero, say,

$$\tau^0=a_1|\sigma^0| \quad \text{and} \quad C_1(x^0)=0.$$

Then we find that $\theta(\sigma)=a_1|\sigma|$,

$$Q(x', \tau, \sigma)=-C_1(x')\tau(|\tau|^2+|\sigma|^2)^{-1/2}, \quad b_{12}(x', \tau, \sigma)=0 \text{ and}$$
$$(b_{21}\tilde{L})(x', \tau, \sigma)=(C_1(x')-C_2(x'))\cdot(\text{nonzero factor}),$$

which implies our assertion.

It is known that for every $x'\in\Gamma$ the constant coefficients problem $(\tilde{P}^0, \tilde{B}_1^0, \tilde{B}_2^0)_{x'}$ is L^2-well posed if and only if (3.1) is valid and $C_1(x')=0$ implies $C_2(x')=0$ (see Theorem 1 of [3] and Lemma 4.1 of [2]). Thus the inequality (3.2) shows that the L^2-well posedness of the variable coefficients problem $(\tilde{P}, \tilde{B}_1, \tilde{B}_2)$ need not follow from that of the constant coefficients problems $(\tilde{P}^0, \tilde{B}_1^0, \tilde{B}_2^0)_{x'}$ for all $x'\in\Gamma$, in contrast with the case of second order or 2×2 systems of first order (see [4], [10]).

The method of considerations used in proving Theorem is applicable to more general cases. The details will be published in Hokkaido Math. J.

The author wishes to express his hearty thanks to Professor T. Shirota for the kind criticisms. He also thanks Dr. R. Agemi for the valuable discussions.

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