147. *The Finite Hilbert Transform on $L_2(0,\pi)$ is a Shift*

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Let v be the finite Hilbert transform on $L_2(0,\pi)$ defined by

$$(V\varphi)(t)=\frac{1}{\pi i}\int_0^\pi \frac{\sin s}{\cos t-\cos s}\varphi(s)ds,$$

where the integral is the Cauchy principal value. In contrast with the development of the spectral theory of a finite Hilbert transform A of the form

$$(Af)(x)=\frac{1}{\pi i}\int_a^b \frac{f(y)}{x-y}dy$$

acting on $L_2(a,b)$, which occurs in airfoil theory, the singular integral operator V on $L_2(0,\pi)$ has not received much attention, while it plays an important role in the theory of singular integral equations (cf. [3]). Let $\varphi_n(t)=\sin nt$ $(n=1,2,\cdots)$ and $\psi_n(t)=\cos nt$ $(n=0,1,2,\cdots)$. Then the sequences $\{\varphi_n\}$ and $\{\psi_n\}$ of vectors are both orthogonal bases in $L_2(0,\pi)$ and as is seen in Hochstadt [3; p. 160], V is an isometry such that

$$V\varphi_n=-i\psi_n \qquad (n=1,2,\cdots).$$

The first object of this paper is to prove the following decisive result:

Theorem. *The finite Hilbert transform V on $L_2(0,\pi)$ is a unilateral shift of multiplicity* 1.

Next we shall indicate that this result actually offers a new technique in the spectral representation theory for the airfoil operator A and enables us to remove somewhat complicated integral calculations involved in the conventional treatments [4] and [7].

1. The proof of the theorem is done independently of the airfoil operator on $L_2(-1,1)$. First observe that the operator V is symmetrizable in the sense of P. Lax [5] (for symmetrizable operators, see also [1] and [9]). Indeed, for a pair of vectors φ,ψ in $L_2(0,\pi)$, we define a new inner product $(\ ,\ )$ by

$$(\varphi,\psi)=\int_0^\pi \varphi(t)\bar{\psi}(t)\sin t\,dt.$$

Then it is obviously bounded on $L_2(0,\pi)$ and from the behavior of V on the basis $\{\varphi_n\}$ it is straightforward to verify that

$$(V\varphi_n,\varphi_m)-(\varphi_n,V\varphi_m)=-i\int_0^\pi \sin(m+n)t\sin t\,dt=0$$

for every n,m. It follows immediately from this that V is self-adjoint with respect to the new inner product.

Proof of Theorem. Since the operator V is an isometry, we decompose it into the direct sum

$$V=V_0\oplus V_1$$

of a unitary operator V_0 and a unilateral shift V_1 acting its reducing subspaces H_0 and H_1, respectively. Then the unitary direct summand V_0 is also symmetrizable. But, as is known, a normal operator can be symmetrizable only if it is self-adjoint (cf. [9]). Thus V_0 is unitary and self-adjoint. This implies that at least one of the values ± 1 must be an eigenvalue of V whenever the direct summand V_0 exists, i.e., $H_0\neq\{0\}$. We shall show, however, that V does not admit either of the values ± 1 as eigenvalue. Suppose that φ is a vector in $L_2(0,\pi)$ such that $V\varphi=\varphi$. Consider the Fourier sine expansion of φ, i.e.,

$$\varphi=\sum_n \lambda_n\varphi_n.$$

Then $V\varphi=\sum_n \lambda_n V\varphi_n=\sum_n \lambda_n(-i)\psi_n$, and hence we have

$$\sum_n (-i)\lambda_n\psi_n=\varphi.$$

Thus it follows that the nth Fourier cosine coefficient of φ is equal to $(-i)\lambda_n$, that is,

$$\frac{2}{\pi}\int_0^\pi \varphi(t)\cos nt\,dt=(-i)\frac{2}{\pi}\int_0^\pi \varphi(t)\sin nt\,dt,$$

so that $\int_0^\pi \varphi(t)e^{int}dt=0$ $(n=1,2,\cdots)$. But $\{e^{int}\}$ $(n=1,2,\cdots)$ is complete in $L_1(0,\pi)$,[1] and so $\varphi(t)=0$ a.e. Next, applying the same argument to a vector φ such that $V\varphi=-\varphi$, we reach $\int_0^\pi \varphi(t)e^{-int}dt=0$ and then we find $\varphi=0$. Therefore, what we have just proved is that each of the values ± 1 is not an eigenvalue of V. Consequently, the unitary direct summand must vanish and hence V is nothing but a unilateral shift.

To see the co-rank of V (called its multiplicity, cf. [2]), it is enough to recall the behavior of V on the φ_n's. Then it is evident that the orthogonal complement of the range of V is one dimensional (indeed, it is the scalar multiples of the identity function). This implies that the co-rank of V is 1. Overall, it turns out that $\{1, V1,\cdots, V^n1,\cdots\}$ is an orthogonal basis in $L_2(0,\pi)$. The proof is now complete.

2. Now let us describe briefly how our result influences on the spectral analysis of the airfoil operator A on $L_2(-1,1)$. Consider the completion K of $L_2(0,\pi)$ with the new norm $|\varphi|=(\varphi,\varphi)^{1/2}$ and let $\hat{V}$ denote the extension of V to K. Then $\hat{V}$ is a self-adjoint contraction.

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1) For the completeness of $\{e^{int}\}$, Levinson's classic [6] states more generally that $\{e^{i\mu_n t}: \mu_n>0\}$ is complete in L_1 over an interval of length L if

$$\liminf_{n\to\infty}\{n/\mu_n\}>L/2\pi.$$

Furthermore, appealing to the mapping $x=\cos t$ of $[0,\pi]$ onto $[-1,1]$, it is easily verified that $\hat{V}$ is unitarily equivalent to the operator A. As mentioned above, the identity function 1 in $L_2(0,\pi)$ is a cyclic vector for V and the new norm $|,|$ is evidently weaker than the usual norm in $L_2(0,\pi)$. Hence 1 is also a cyclic vector for $\hat{V}$. From this we immediately reach a conclusion that A *is a self-adjoint contraction having a cyclic vector*, (compare with the proof in [7]). Consequently, applying the spectral representation theorem for a general normal opertor to A, we can assert that A is unitarily equivalent to the multiplication operator M:

$$(Mh)(\lambda)=\lambda h(\lambda)$$

on $L_2(\Omega,\mu)$, where Ω is the spectrum and u is a regular Borel measure on Ω induced by the spectral measure for A. But the estimate of the measure μ and the fact that Ω is the interval $[-1,1]$, which are nowadays well known, don't follow directly from our result. Indeed, we can introduce various different inner products $(\ ,\ )_\alpha(\alpha\in\Lambda)$ on $L_2(0,\pi)$ which make the shift-adjoint, i.e.,

$$(V\varphi,\psi)_\alpha=(\varphi,V\psi)_\alpha,$$

and the spectrum of a self-adjoint contraction $\hat{V}_\alpha$, the extension of V with respect to $(\ ,\ )_\alpha$, is not necessarily the interval $[-1,1]$. It should be pointed out, however, that the spectrum of $\hat{V}$ and its spectral measure (which are the same as Ω and μ, respectively) are estimated, without any difficulties, from two general theorems on commutators due to C. Putnam [8; Theorem 2.2.1 and Theorem 2.2.4]. Thus, according to the linkages between the unilateral shift and finite Hilbert transforms, one can complete the spectral representation for the airfoil operator A on $L_2(-1,1)$ *by purely operator theoretic arguments*. In the same manner, the spectral representation for the airfoil operator on $L_2(a,b)$ (b may be $+\infty$) may be obtained via an appropriate mapping of $[0,\pi]$ onto $[a,b]$.

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