7. *Quadruply-Transitive Permutation Groups Whose Four-Point Stabilizer is a Frobenius Group*

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1. Introduction. In this paper we shall prove the following result.

Theorem. *Let G be a 4-fold transitive group on $\Omega=\{1,2,\cdots,n\}$. If the stabilizer of four points in G is a Frobenius group, then G is one of the following groups*: S_7, A_8 *or* M_{23}.

We shall use the same notations as in [1].

2. Proof of the theorem. Let K be the Frobenius kernel of G_{1234} and H a Frobenius complement of G_{1234}.

By a theorem of M. Hall, the order of G_{1234} is even.

Let P be a Sylow 2-subgroup of G_{1234}. Then $P\neq 1$. If P is isomorphic to a subgroup of H, then G is S_7 by Theorem 1 in [2]. Hence we may assume that P is contained in K. Thus P is a normal subgroup of G_{1234}.

By [1; IV] and Lemma 1 in [1; II], $|I(G_{1234})|=4$ and $|I(P)|=4,5,6,7$ or 11. If $|I(P)|\geqq 6$, then G is M_{23} by [1; VIII, IX, XI]. If $|I(P)|=5$, then $|I(G_{1234})|=5$, which is a contradiction. Hereafter we assume $|I(P)|=4$, and so, that n is an even integer.

If P is semiregular on $\Omega-I(P)$ or P is abelian, then G is A_8 by [1; VII, X]. From now on, we shall examine the case where P is neither semiregular on $\Omega-I(P)$ nor abelian, and prove eventually that this case does not arise.

Let R be a Sylow 3-subgroup of G_{1234}. By [1; XIII] and [3], R is a nonidentity group and $[R,P]\neq 1$. If R is contained in K, then $[R,P]=1$, which is a contradiction. Hence R must be contained in a conjugate of H.

Let r be an element of order three of R. Then r is an element of order three acting fixed point free on $P-\{1\}$. Hence by [4], the nilpotency class of P is two.

By Theorem A in [5], G_{123} has either (1) an abelian normal subgroup $\neq 1$, or (2) a unique minimal normal subgroup, and this minimal normal subgroup is simple. In the case (1), G must be S_7 or M_{23} by [6], in contradiction to our present assumption $|I(P)|=4$. We shall now consider the case (2). Let N be the minimal normal subgroup of G_{123}. It is easily seen that G_{123} is contained in $Aut(N)$.

Let S be a Sylow 2-subgroup of N. Since n is even, we may assume that S is contained in P. Hence the nilpotency class of S is one or two.

By [7] and [8], N is one of the following groups: $PSL(2, 2^m)$ $(m>1)$, $PSL(2, q)$ ($q\equiv 3$ or $5 \pmod 8$, $q>3$), Ja, Ree group, $PSL(2, q)$ ($q\equiv 7$ or $9 \pmod{16}$), A_7, $Sz(2^{2m+1})$ $(m\geqq 1)$, $U_3(2^m)$ $(=PSU(3, 2^m))$ $(m\geqq 2)$, $L_3(2^m)$ $(=PSL(3, 2^m))$ $(m\geqq 2)$ or $Sp(4, 2^m)$ $(m\geqq 2)$.

Suppose N is Ja, Ree group, A_7 or $PSL(2, q)$ ($q\equiv 7$ or $9 \pmod{16}$). Then S is of order 8. As r normalizes P and N, r normalizes S, and r acts fixed point free on $S-\{1\}$. Hence $3\,|\,7$, which is a contradiction.

Suppose N is $PSL(2, 2^m)$ $(m>1)$. Then we may assume that

$$S=\left\{\begin{pmatrix}1 & 0\\ x & 1\end{pmatrix}\middle|\, x\in GF(2^m)\right\}.$$

Since $3\,|\,2^m-1$, $3\nmid 2^{m+1}-1$. Thus there must be an element of order 4 in G_{123} which is a field automorphism of $GF(2^m)$. Let a be that element. Then

$$Z(\langle a\rangle \ltimes S)=\left\{\begin{pmatrix}1 & 0\\ x & 1\end{pmatrix}\middle|\, x\in GF(2^{m/4})\right\}.$$

Hence the nilpotency class of $\langle a\rangle\ltimes S$ is greater than two, which is a contradiction.

Suppose N is $PSL(2, q)$ ($q\equiv 3$ or $5 \pmod 8$, $q>3$). Then G_{123} is contained in $P\Gamma L(2, q)$. Hence P is a dihedral group or a semidihedral group. On the other hand the order of P is not less than 16. Hence $P/Z(P)$ is a non-abelian group, which is a contradiction.

Suppose N is $Sz(2^{2m+1})$ $(m\geqq 1)$, $U_3(2^m)$ $(m\geqq 2)$ or $L_3(2^m)$ $(m\geqq 2)$. Then it is easily seen that $P=S$. Since $|I(P)|=4$ and $N\leqq G_{123}$, $N_N(S)=N_N(P)\leqq G_{1234}$. Let i be an involution of $Z(S)$. As $Sz(2^{2m+1})$, $U_3(2^m)$ and $L_3(2^m)$ are C-groups, we have $C_N(i)\leqq N_N(S)$. Since $C_N(i)$ is contained in G_{1234}, $C_N(i)$ is a nilpotent group. Hence N must be $Sz(2^{2m+1})$. On the other hand $\mathrm{Syl}_2(G_{123})=\mathrm{Syl}_2(N)$ and P is not semiregular on $\Omega-I(P)$. Hence the Sylow 2-subgroups are not disjoint, which is a contradiction.

Suppose N is $Sp(4, q)$ ($q=2^m$, $m\geqq 2$). Then it is easily seen that $P=S$. We may assume that $N=\{y\in GL(4, q)\,|\, y^t\cdot j\cdot y=j\}$. In this case

$$j=\begin{pmatrix}0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\end{pmatrix}.$$

We may assume that

$$S=\left\{\begin{pmatrix}1 & d & e & f\\ 0 & 1 & g & e+dg\\ 0 & 0 & 1 & d\\ 0 & 0 & 0 & 1\end{pmatrix}\middle|\, d, e, f, g\in GF(q)\right\}.$$

Since $P=S$, we have $N_N(S)\leqq G_{1234}$. Let c be a generator of the cyclic group $GF(q)^*$. Let

$$u=\begin{pmatrix}c & 0 & 0 & 0\\ 0 & c^{-1} & 0 & 0\\ 0 & 0 & c & 0\\ 0 & 0 & 0 & c^{-1}\end{pmatrix},\qquad v=\begin{pmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$

and

$$w=\begin{pmatrix}1 & 0 & 1 & 1\\ 0 & 1 & 1 & 1\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}.$$

Then u normalizes S and u is an odd order element of G_{1234}. v and w are involutions of P, and $vu=uv$ but $wu\neq uw$. This is a contradiction.

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