3. Abelian Groups and N-Semigroups. II

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1. Introduction. This note takes its name from the paper [4] by Takayuki Tamura. In that paper Tamura shows the following result:

Theorem 1.1. *Let K be an Abelian group and A be the group of integers under addition. If G is an Abelian extension of A by K with respect to factor system $f: K\times K \to A$, then there exists a factor system g such that*

(i) *$g(\alpha, \beta) \geq 0$ for all α, β in K*

(ii) *g is equivalent to f.*

There needs to be a slight change in the proof. Define a new function δ' by $\delta'(\varepsilon)=0$ and $\delta'(\alpha)=\delta(\alpha)$ if $\alpha\neq\varepsilon$. Let $g(\alpha, \beta)=f(\alpha, \beta)+\delta'(\alpha)+\delta'(\beta)-\delta'(\alpha\beta)$.

In his paper Tamura asks if A in Theorem 1.1 can be replaced by an ordered Abelian group. We shall show that A can be replaced by any subgroup of the additive reals. Alternatively we shall show that A can be an Archimedean ordered Abelian group, as an Archimedean ordered Abelian group is isomorphic to a real semigroup.

2. Preliminary results. Let A be a subgroup of the reals under addition. Let G be an Abelian group containing A. Let S be an N-subsemigroup (see [4]) of G which contains $A^+=\{x\in A: x>0\}$ such that G is the quotient group of S. We call A^+ positive cone of A. Let $G=\bigcup_{\xi\in G/A} A_\xi$ be the decomposition of G into cosets modulo A. Let $x\in A_\xi$, some arbitrary coset of G, then $x=bc^{-1}$ for some $b, c\in S$. Let $a\in A^+\subset S$. As S is Archimedean there exists positive integer m and some $d\in S$ such that $cd=a^m$. Thus $xc=b$ implies $xa^m=xcd=bd\in S$. Note that as $x\in A_\xi$ and as $a^m\in A$ we have $xa^m\in A_\xi$ and so $S\cap A_\xi\neq\emptyset$.

Proposition 2.1. *Let A be a subgroup of the reals under addition and G be an Abelian group containing A. Let S be an N-subsemigroup of G which contains A^+. The following are equivalent:*

(i) *G is the quotient group of S.*

(ii) *$G=AS$.*

(iii) *S intersects each congruence class of G modulo A.*

Proof. We have shown that (i) implies (iii). For any commutative cancellative semigroup T, we let $Q(T)$ denote the quotient group of T. If $G=AS$ then as $A^+\subset S$ we have $A=Q(A^+)\subset Q(S)$ and so $G=AS$

$\subset Q(S)$. It follows that (ii) implies (i). Suppose S intersects each congruence class of G modulo A. Let A_ξ be an arbitrary congruence class of G modulo A and let $x \in S \cap A_\xi$. Note that $A_\xi = Ax \subset AS$. This is true for each $\xi \in G/A$ and so $G = AS$. We thus have (iii) implies (ii).

For any Abelian group T we shall let $D(T)$ denote the divisible hull of T.

Proposition 2.2. *Let G be an Abelian group which contains A, a subgroup of the additive reals. There exists an N-subsemigroup S of G containing A^+ such that G is the quotient group of S.*

Proof. As the additive group of reals is divisible we have that $D(A)$ is a subgroup of the reals. It is well known from group theory [2] that a divisible subgroup of a group is a direct summand and so $D(G) = D(A) \oplus L$ for some Abelian group L. Let $S^* = D(A)^+ \oplus L$. S^* is an N-semigroup which contains A^+. Let $S = S^* \cap G$. S contains A^+ as $A^+ \subset S^*$ and $A^+ \subset G$. Let $\pi : D(G) \to D(A)$ be the projection homomorphism. Let $a \in A^+ \subset D(G)$, then $\pi(a) > 0$. Let $x \in G$. There exists a positive integer n such that $n\pi(a) + \pi(x) > 0$ and so $\pi(na + x) > 0$. Hence $na + x \in G \cap (D(A)^+ \oplus L)$ implying that $na + x \in S$. Hence $G \subset A + S$ and so $G = A + S$. By Proposition 2.1 G is the quotient group of S. Let $x, y \in S$. As S^* is Archimedean we have $mx = y + z$ for some $z \in S^*$ and some positive integer m. As $x, y \in G$ we have $z \in G$. As $z \in S^* \cap G = S$, S is Archimedean. S is thus an N-subsemigroup of G, containing A^+, whose quotient group is G.

Remark 2.3. In Proposition 2.2, any N-subsemigroup S of G containing A^+ satisfies $S \cap A = A^+$.

Proof. This follows as S is idempotent free.

3. Applications to Abelian group theory.

Theorem 3.1. *Let K be an Abelian group and A be a subgroup of the reals under addition. If G is an Abelian extension of A by K with respect to a factor system $f : K \times K \to A$, then there exists a factor system g such that*

(i) *$g(\alpha, \beta) \geq 0$ for all $\alpha, \beta \in K$ and*

(ii) *g is equivalent to f.*

Proof. By the assumption, let $G = \{(m, \alpha) : \alpha \in K,\ m \in A\}$ in which $(m, \alpha)(n, \beta) = (m + n + f(\alpha, \beta), \alpha\beta)$. Let e be the identity of K. We identify A^+ and $\{(x, e) : x \in A^+\}$. By Proposition 2.2 there is an N-semigroup S containing A^+ such that G is the quotient group of S. By Remark 2.3 $S \cap A = A^+$. Let $\xi \in K$. Suppose there exists a collection $\{(x_n, \xi)\}_{n=1}^{\infty}$ of elements of S such that $x_n \to -\infty$. Let $(y, \xi^{-1}) \in S$. Note that such an element exists as S intersects each congruence class of G modulo A. For each positive integer n, $(x_n, \xi)(y, \xi^{-1}) = (x_n + y + f(\xi, \xi^{-1}), e) \in S \cap A = A^+$. This is a contradiction as $x_n + y + f(\xi, \xi^{-1}) \to -\infty$. For each $\alpha \in K$ we can thus define $\sigma(\alpha) = \inf\{x : (x, \alpha) \in S\}$. Note that $\sigma(e) \neq 0$ if and only if A is isomorphic to the group of integers. This case has been treated by Tamura. Thus we may assume that A is not isomorphic to the group of integers and so $\sigma(e)=0$. Let $\{(x_n, \alpha)\}$, $\{(y_n, \beta)\}$ be subsets of S such that $x_n \to \sigma(\alpha)$ and $y_n \to \sigma(\beta)$. Then for each positive integer n, $(x_n + y_n + f(\alpha, \beta), \alpha\beta) \in S$. It follows that for each positive integer n we have $x_n + y_n + f(\alpha, \beta) \geq \sigma(\alpha\beta)$ and so $\sigma(\alpha) + \sigma(\beta) + f(\alpha, \beta) \geq \sigma(\alpha\beta)$. Let $g(\alpha, \beta) = f(\alpha, \beta) + \sigma(\alpha) + \sigma(\beta) - \sigma(\alpha\beta)$ for every $\alpha, \beta \in K$. We see that g is a factor system which is equivalent to f and $g(\alpha, \beta) \geq 0$ for all $\alpha, \beta \in K$.

References

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