# 32. On Multivalent Functions in Multiply Connected Domains. I

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**1. Introduction.** E. Rengel [3] derived many results on univalent functions in the multiply connected, representative domains (defined hereafter) by means of the so-called Rengel's inequality. In this paper we shall deal with multivalent functions in multiply connected domains by means of the fundamental inequalities based on the extremal length method which are extensions of Rengel's inequality (cf. [2] or [4]).

We shall first define the $n$-ply connected, representative domains as follows.

$D_1$: an annulus, $(0<)\ r_1<|z|<r_2\ (<\infty)$ with $(n-2)$ circular arc slits centered at the origin.

$D_2$: an annulus, $(0<)\ r_1<|z|<r_2\ (<\infty)$ with $(n-2)$ radial slits emanating from the origin.

$D_3$: the unit circle with $(n-1)$ circular arc slits centered at the origin.

$D_4$: the unit circle with $(n-1)$ radial slits emanating from the origin.

$D_5$: the whole plane with $n$ circular arc slits centered at the origin.

$D_6$: the whole plane with $n$ radial slits emanating from the origin.

We shall define circumferentially mean $p$-valent functions in a domain $D$, according to Biernacki (cf. Hayman [1]).

Let $n(R,\Phi)$ denote the number of roots of the equation $f(z)=w=\mathrm{Re}^{i\Phi}$. We define $p(R)$ as follows.

$$p(R)=\frac{1}{2\pi}\int_0^{2\pi} n(R,\Phi)d\Phi \qquad (0\le R<\infty). \tag{1.1}$$

If $p(R)\le p$ $(0\le R<\infty)$, $f(z)$ is called "circumferentially mean $p$-valent". In this paper we assume that $p$ is a positive integer.

**2. Fundamental inequalities. Theorem 2.1.** *Let $f(z)$ be single-valued, regular, circumferentially mean $p$-valent in $D_1$ and satisfy the condition $\int_C |d\arg f(z)|\ge 2\pi p(C: |z|=r\ (r_1<r<r_2))$ where the circle $C$ does not contain any circular slit of $D_1$. Then we have the following inequality*

$$(2.1)\qquad \frac{R_2}{R_1}\geq\left(\frac{r_2}{r_1}\right)^p \qquad \left(R_1\equiv \inf_{z\in D_1}|f(z)|,\ R_2=\sup_{z\in D_1}|f(z)|\right).$$

*Equality holds only when* $f(z)=cz^p$ ($c$: *an arbitrary constant*).

**Proof.** We shall introduce a weight function $\rho(z)=|f'(z)|/(2\pi|f(z)|)$. Then

$$(2.2)\quad \int_C \rho(z)rd\varphi=\frac{1}{2\pi}\int_C|d\log f(z)|\geq\frac{1}{2\pi}\int_C|d\arg f(z)|\geq p \quad (\varphi=\arg z).$$

Therefore, on every circle $C$ we have $\int_0^{2\pi}\rho d\varphi\geq p/r$. Hence, considering $\iint_{D_1}(\rho-p/2\pi r)^2 rdrd\varphi\geq 0$, we have

$$(2.3)\qquad \iint_{D_1}\rho^2 rdrd\varphi\geq\frac{p^2}{2\pi}\log\frac{r_2}{r_1}.$$

Here $(2\pi)^2\iint_{D_1}\rho^2 rdrd\varphi$ means the logarithmic area of the image domain $D_1^*$ of $D_1$. Then $(2\pi)^2\iint_{D_1}\rho^2 rdrd\varphi=\iint_{D_1^*}(n(R,\Phi)/R)dRd\Phi(w=f(z)=\mathrm{Re}^{i\Phi})$. On the other hand

$$(2.4)\quad \iint_{D_1^*}\frac{n(R,\Phi)}{R}dRd\Phi=\int_{R_1}^{R_2}\frac{dR}{R}\int_0^{2\pi}n(R,\Phi)d\Phi\leq 2\pi p\log\frac{R_2}{R_1}.$$

**Theorem 2.2.** *Let $f(z)$ be single-valued, regular, circumferentially mean p-valent in $D_2$. Let $M=\{\gamma_\varphi\}$ denote the family of the segments, $r_1<|z|<r_2$, $\arg z=\varphi$ $(0\leq\varphi<2\pi)$ which do not contain any radial slit of $D_2$. Then we have the following inequality,*

$$(2.5)\qquad p\log\frac{R_2}{R_1}\log\frac{r_2}{r_1}\geq A^2,$$

*where* $\inf_{\gamma_\varphi\in M}\int_{r_1}^{r_2}|f'(z)|/|f(z)|\,dr\equiv A$, $R_1=\inf_{z\in D_2}|f(z)|$, $R_2\equiv\sup_{z\in D_2}|f(z)|$. *Equality holds only when* $f(z)=cz^p$ ($c$: *an arbitrary constant*).

**Proof.** Similarly as in Theorem 2.1, we shall do the proof.

$$(2.6)\qquad \iint_{D_2}\left(\rho-\left(2\pi\log\frac{r_2}{r_1}\right)^{-1}\frac{A}{r}\right)^2 rdrd\varphi\geq 0$$

$$\left(\rho=\frac{1}{2\pi}\left|\frac{f'(z)}{f(z)}\right|,\ z=re^{i\varphi}\right).$$

Since $\int_{\gamma\varphi}\rho dr\geq A/2\pi$, we have

$$(2.7)\qquad \iint_{D_2}\rho^2 rdrd\varphi\geq A^2/(2\pi\log(r_2/r_1)).$$

Hence, we can derive (2.5) by means of (2.4).

**3. Applications of fundamental inequalities.** **Theorem 3.1.** *Let $f(z)$ be single-valued, circumferentially mean p-valent, and $|f(z)|<1$ in $D_3$. Moreover let $\int_{\gamma\nu}darg\,f(z)=0$ $(\nu=1,2,\cdots,n-1)$ along every curve*

*$\gamma_\nu$ in $D_3$ which is sufficiently near to each arc slit $S_\nu$ and encloses it simply, and $f(z)$ be expanded in a neighborhood of the origin as follows:*

$$f(z)=a_p z^p + a_{p+1} z^{p+1} + \cdots.$$

*Then*

$$|a_p| \leq 1. \tag{3.1}$$

*Equality holds only when $f(z)=cz^p$ ($|c|=1$).*

**Proof.** $f(z)$ has a zero point of order $p$ at the origin and $f(z)$ is circumferentially mean $p$-valent in $D_3$. Hence $f(z)$ has no zero point except at $z=0$. Therefore we have $\int_{|z|=r} d \arg f(z) - \sum_{\nu=1}^{k} \int_{\gamma_\nu} d \arg f(z) = 2\pi p$ for every circle $|z|=r$ $(0<r<1)$ $(k=1,2,\cdots,n-1)$ where each $\gamma_\nu$ satisfies the condition in Theorem 3.1. Hence $\int_{|z|=r} d \arg f(z)=2\pi p$.

Let $\delta(\varepsilon)$ denote the nearest distance from the origin to the image of a small circle $|z|=\varepsilon$ by $w=f(z)$. Then we have $\lim_{\varepsilon \to 0} \delta(\varepsilon)/\varepsilon^p = a_p$.

On the other hand, applying Theorem 2.1 for the image of the domain obtained by omitting a circle $|z| \leq \varepsilon$ from $D_3$, we have $1/\varepsilon^p \leq 1/\delta(\varepsilon)$. Hence we have $|a_p| \leq 1$.

**Theorem 3.2.** *Let $f(z)$ be single-valued, regular, circumferentially mean p-valent and $|f(z)|<1$ in $D_4$. Moreover let in a neighborhood of the origin, $f(z)=a_p z^p + a_{p+1} z^{p+1} + \cdots$. Then*

$$|a_p| \geq m^2 \qquad \left(m=\min_{|z|=1} |f(z)|\right). \tag{3.2}$$

*Equality holds only when $f(z)=cz^p$ ($|c|=1$).*

**Proof.** Let $D_4^*$ denote the domain obtained by omitting a closed small circle $|z| \leq \varepsilon$ from $D_4$. Let $\gamma_\varphi$ denote a radial segment, $\varepsilon<|z|<1$, $\arg z=\varphi$ $(0 \leq \varphi < 2\pi)$ which does not contain any radial slit of $D_4$. Let $\delta(\varepsilon)$ or $\delta^*(\varepsilon)$ denote respectively the longest and nearest distance from the origin to the image of a circle $|z|=\varepsilon$ by $w=f(z)$. Then $\lim_{\varepsilon \to 0} \delta(\varepsilon)/\varepsilon^p = \lim_{\varepsilon \to 0} \delta^*(\varepsilon)/\varepsilon^p = |a_p|$. On the other hand

$$\inf L_\varphi \geq \log \frac{m}{\delta(\varepsilon)} \qquad \left(L_\varphi = \int_{\gamma_\varphi} \left| \frac{f'(z)}{f(z)} \right| dr\right). \tag{3.3}$$

Applying Theorem 2.2 for $D_4^*$, we have $(\log m/\delta(\varepsilon))^2 \leq p \log(1/\delta^*(\varepsilon)) \times \log(1/\varepsilon)$, that is, $(\log m\varepsilon^p/\delta(\varepsilon) - p \log \varepsilon)^2 \leq -p(\log \varepsilon^p/\delta^*(\varepsilon) - p \log \varepsilon) \log \varepsilon$. Hence we can derive (3.2).

We can prove the following lemma, by means of argument principle.

**Lemma 3.1.** Let $f(z)$ be single-valued, regular except for the pole at $\infty$, circumferentially mean $p$-valent in $D_5$ and expanded in a neighborhood of the origin, $f(z)=z^p + a_{p+1} z^{p+1} + \cdots$. Moreover let

$\int_{\gamma_\nu} d\arg f(z)=0$ ($\nu=1, 2, \cdots, n$) for every simple closed curve $\gamma_\nu$ which is sufficiently near to each circular slit $S_\nu$ and encloses $S_\nu$. Then $f(z)$ has only a pole of order $p$ at $z=\infty$.

We can easily prove the following by means of Lemma 3.1 and Theorem 2.1, considering the method of cutting a neighborhood of $z=\infty$.

**Theorem 3.3.** *Let $f(z)$ satisfy the same condition as mentioned in Lemma 3.1. Then*

$$\lim_{z\to\infty}\left|\frac{f(z)}{z^p}\right|\geq 1. \tag{3.4}$$

*Equality holds only when $f(z)=z^p$.*

We can also prove the following by Theorem 2.2 similarly.

**Theorem 3.4.** *Let $f(z)$ be single-valued, regular, except at $z=\infty$, circumferentially mean $p$-valent in $D_6$ and $f(z)=z^p\sum_{n=0}^{\infty} b_n z^{-n}$ ($b_0=1$) in a neighborhood of $z=\infty$. Moreover let $f(z)=a_p z^p+a_{p+1}z^{p+1}+\cdots$, in a neighborhood of the origin. Then*

$$|a_p|\geq 1. \tag{3.5}$$

*Equality holds only when $f(z)=z^p$.*

## References

[1] W. K. Hayman: Multivalent Functions. Cambridge Tracts, No. 48 (1958).
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[3] E. Rengel: Über einige Schlitztheoreme der konformen Abbildung. Schr. Math. Seminars. u. Inst. f. angew. Math. Univ. Berlin, 7, 141–162 (1933).
[4] M. Tsuji: Potential Theory in Modern Function Theory. Chelsea Pub. Co., New York.