29. On Deformations of Compactifiable Complex Manifolds

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In this note we shall extend the deformation theory of compact complex manifolds to compactifiable ones defined below.

1. We fix our notation.

$\bar{X}$: a compact complex manifold,

$\bar{D}$: a closed analytic subset of $\bar{X}$ (not necessarily reduced),

$X := \bar{X} - \bar{D}$,

$I_{\bar{D}}$: the ideal sheaf of $\bar{D}$ in $\mathcal{O}_{\bar{X}}$,

$T_{\bar{X}}(\log \bar{D})$: the subsheaf of the tangent sheaf $T_{\bar{X}}$ consisting of derivations of $\mathcal{O}_{\bar{X}}$ which send $I_{\bar{D}}$ into itself.

$\bar{D}$ is said to be of simple normal crossing if (1) $\bar{D}=\bigcup_{i=1}^{h} \bar{D}_i$ where the $\bar{D}_i$ are complex submanifolds of $\bar{X}$, and (2) for each $p \in \bar{X}$, there exist a neighborhood U of p and a system of local coordinates $\{z_1, \cdots, z_n\}$ on U such that $\bar{D}_i=\{z_{r_i+1}=\cdots=z_{r_{i+1}}=0\}$ for $1 \leqslant i \leqslant h$, where the r_i are integers such that $-1 \leqslant i \leqslant n$ and $r_i \leqslant r_j$ if $i \leqslant j$ and we put $z_0=1$ by convention. In that case $\bar{X}$ is called a non-singular compactification of X and $(X, \bar{X}, \bar{D})$ is called *a non-singular triple*. For a fixed X, a bimeromorphic equivalence class $\mathfrak{m}$ of non-singular compactifications of X is called *a meromorphic structure* of X. A pair $(X, \mathfrak{m})$ is called a *compactifiable complex manifold.*

By a family of *logarithmic deformations* of a non-singular triple we mean a 7-tuple $\mathcal{F}=(\mathfrak{X}, \bar{\mathfrak{X}}, \bar{\mathfrak{D}}, \pi, S, s_0, \bar{\psi})$ such that (1) $\pi: \bar{\mathfrak{X}} \to S$ is a proper smooth morphism of (not necessarily reduced) complex spaces $\bar{\mathfrak{X}}$ and S, (2) $\bar{\mathfrak{D}}$ is a closed analytic subset of $\bar{\mathfrak{X}}$ and $\mathfrak{X}=\bar{\mathfrak{X}}-\bar{\mathfrak{D}}$, (3) $\bar{\psi}: \bar{X} \to \pi^{-1}(s_0)$ is an isomorphism such that $\bar{\psi}(X)=\pi^{-1}(s_0) \cap \mathfrak{X}$, and (4) π is locally a projection of a product space as well as the restriction of it to $\bar{\mathfrak{D}}$. A family of *compactifiable deformations* of a compactifiable complex manifold $(X, \mathfrak{m})$ is a 5-tuple $(\mathfrak{X}, \pi, S, s_0, \psi)$ obtained from the 7-tuple above.

Theorem 1. *We have the following exact sequences:*

(1) $$0 \longrightarrow T_{\bar{X}}(-\bar{D}) \longrightarrow T_{\bar{X}}(\log \bar{D}) \longrightarrow T_{\bar{D}} \longrightarrow 0$$

where $T_{\bar{D}}$ is the sheaf of derivations $\mathrm{Der}_{\mathcal{O}_{\bar{D}}}(\mathcal{O}_{\bar{D}}, \mathcal{O}_{\bar{D}})$.

(2) $$0 \longrightarrow T_{\bar{X}}(\log \bar{D}) \longrightarrow T_{\bar{X}} \longrightarrow N_{\bar{D}} \longrightarrow 0$$

where $N_{\bar{D}}=\mathrm{Coker}\,(T_{\bar{D}} \to T_{\bar{X}} \otimes_{\mathcal{O}_{\bar{X}}} \mathcal{O}_{\bar{D}})$.

In case where $\bar{D}$ is of simple normal crossing, we see easily $N_{\bar{D}} = \bigoplus_{i=1}^{h} N_{\bar{D}_i}$ where the $N_{\bar{D}_i}$ are the normal sheaves of the $\bar{D}_i$ in $\bar{X}$. Theorem 1 enables us to calculate several cohomology groups.

2. In this section $(X, \bar{X}, \bar{D})$ is a non-singular triple (thus $\bar{D}$ is always assumed to be of simple normal crossing.

Theorem 2. *We have a semi-universal family of logarithmic deformations of* $(X, \bar{X}, \bar{D})$.

First we reduce to the case where $\bar{D}$ is a divisor by Theorem 9.1 of [3]. In that case $T_{\bar{X}}(\log \bar{D})$ is a locally free sheaf on $\bar{X}$ and the harmonic integral theory is available. Therefore the proof of Theorem 2 can be carried out in an almost parallel way to [6] or [1].

An *admissible center* (resp. a *canonical center*) C on $\bar{X}$ with respect to $\bar{D}$ is a closed submanifold on $\bar{X}$ of codimension at least 2 contained in $\bar{D}$ satisfying the following condition: For each $p \in C$ there exist a neighborhood U of p in X and a closed subset $E_U = \bigcup_{i=1}^{k} E_{U,i}$ of simple normal crossing in U such that $\bar{D} \cap U = \bigcup_{i=1}^{h} E_{U,i}$ $(h \leqslant k)$ (resp. $h=k$) and $C \cap U = \bigcap_{i \in N} E_{U,i}$ for some $N \subset [1, k]$.

Theorem 3. *Let $\bar{X}'$ be a monoidal transform of $\bar{X}$ with an admissible center C with respect to $\bar{D}$ and let $\bar{D}' = \mathrm{red}\,(f^{-1}(\bar{D}))$ where $f: \bar{X}' \to \bar{X}$ is the natural morphism. Then $\bar{D}'$ is a closed analytic subset of simple normal crossing in $\bar{X}'$ and*

$$\boldsymbol{R}f_* T_{\bar{X}'}(\log \bar{D}') = T_{\bar{X}}(\log \bar{D})(\log C),$$

where the right hand side is the intersection of $T_{\bar{X}}(\log \bar{D})$ and $T_{\bar{X}}(\log C)$ in $T_{\bar{X}}$. In particular if C is a canonical center

$$\boldsymbol{R}f_* T_{\bar{X}'}(\log \bar{D}') = T_{\bar{X}}(\log \bar{D}).$$

Theorem 4. *Let $(X, \bar{X}_1, \bar{D}_1)$ and $(X, \bar{X}_2, \bar{D}_2)$ be two non-singular triples and $f: \bar{X}_1 \to \bar{X}_2$ a morphism such that the following diagram*

$$\begin{array}{ccc} & & \bar{X}_1 \\ & \nearrow & \\ X & & \downarrow f \\ & \searrow & \\ & & \bar{X}_2 \end{array}$$

is commutative. Then for an arbitrary family $(\mathfrak{X}, \bar{\mathfrak{X}}_1, \mathfrak{D}_1, \bar{\pi}_1, S_1, s_0, \bar{\Psi}_1)$ of logarithmic deformations of $(X, \bar{X}_1, \bar{D}_1)$, there exist an open neighborhood S_2 of s_0 in S_1 and a family $(\mathfrak{X}|_{S_2}, \bar{\mathfrak{X}}_2, \mathfrak{D}_2, \bar{\pi}_2, S_2, s_0, \bar{\Psi}_2)$ of logarithmic deformations of $(X, \bar{X}_1, \bar{D}_1)$ and a morphism $\mathfrak{f}: \bar{\mathfrak{X}}_1|_{S_2} \to \bar{\mathfrak{X}}_2$ such that $\mathfrak{f} \circ \bar{\Psi}_1 = \bar{\Psi}_2 \circ f$ on $\bar{X}_1$ and the following diagram

$$\begin{array}{ccccc} \bar{\mathfrak{X}}_1|_{S_2} & & \xrightarrow{\ \mathfrak{f}\ } & & \bar{\mathfrak{X}}_2 \\ & \nwarrow & & \nearrow & \\ & & \mathfrak{X}|_{S_2} & & \\ & {\scriptstyle \bar{\pi}_1|_{S_2}} \searrow & & \swarrow {\scriptstyle \bar{\pi}_2} & \\ & & S_2 & & \end{array}$$

is commutative.

By Theorem 9.1 of [3], we reduce to the case where $\bar{D}_1$ and $\bar{D}_2$ are divisors. Then by Theorem 8.1 of loc. cit., we prove the theorem.

Theorem 5. *With the same notation as Theorem* 4, *we assume furthermore that* $\bar{D}_1$ *and* $\bar{D}_2$ *are divisors and* f *is allowable in the sense of toroidal embeddings* [5]. *Then the set of germs of families of compactifiable deformations of* X *induced by the families of logarithmic deformations of* $(X, \bar{X}_1, \bar{D}_1)$ *coincides with that of* $(X, \bar{X}_2, \bar{D}_2)$.

From Theorems 2, 3 and 4, we obtain the following four theorems.

Theorem 6. *Let* $\mathcal{F}=(\mathfrak{X}, \bar{\mathfrak{X}}, \bar{\mathfrak{D}}, \bar{\pi}, S, s_0, \bar{\Psi})$ *be a family of logarithmic deformations of* $(X, \bar{X}, \bar{D})$. *If the Kodaira-Spencer map* $\rho: T_{S,s_0} \to H^1(\bar{X}, T_{\bar{X}}(\log \bar{D}))$ *is surjective and* S *is regular then* $\mathcal{F}$ *is versal at* s_0.

Theorem 7. *If* $H^1(\bar{X}, T_{\bar{X}}(\log \bar{D}))=0$, *then* $(X, \bar{X}, \bar{D})$ *is rigid.*

Theorem 8. *Let* $\mathcal{F}=(\mathfrak{X}, \bar{\mathfrak{X}}, \bar{\mathfrak{D}}, \bar{\pi}, S, s_0, \bar{\Psi})$ *be a family of logarithmic deformations of* $(X, \bar{X}, \bar{D})$. *Assume that* $\dim H^1(\bar{X}_t, T_{\bar{X}_t}(\log \bar{D}_t))$ *is constant and* $\rho_t: T_{S,t} \to H^1(\bar{X}_t, T_{\bar{X}_t}(\log \bar{D}_t))$ *is zero for all* $t \in S$, *where* $\bar{X}_t = \bar{\pi}^{-1}(t)$ *and* $\bar{D}_t = \bar{\mathfrak{D}} \cap \bar{X}_t$. *Then* $\mathcal{F}$ *is a trivial family near* s_0.

Theorem 9. *If* $H^2(\bar{X}, T_{\bar{X}}(\log \bar{D}))=0$, *then there exists a semi-universal family* $(\mathfrak{X}, \bar{\mathfrak{X}}, \bar{\mathfrak{D}}, \bar{\pi}, S, s_0, \bar{\Psi})$ *of logarithmic deformations of* $(X, \bar{X}, \bar{D})$ *such that* S *is regular.*

By a *semi-complex torus* we mean a compactifiable complex manifold whose underlying complex manifold is a topologically trivial principal $(\boldsymbol{C}^*)^d$-bundle over a complex torus and whose meromorphic structure is a natural one.

Theorem 10. *A small compactifiable deformation of a semi-complex torus is again a semi-complex torus.*

To prove the theorem we use the quasi-Albanese map extended for compactifiable complex manifolds (cf. [4]).

3. The technique developed so far is applicable to the study of *equi-singular deformations* of an isolated singularity. Let X be a Stein neighborhood of the origin in $\boldsymbol{C}^n$ and D a reduced closed analytic subset of X such that Sing $D=\{0\}$. By [2] we can find a morphism $f: \tilde{X} \to X$ by successive permissible monoidal transformations such that $\tilde{D}=\text{red}\,(f^{-1}(D))$ is of simple normal crossing.

Theorem 11. *There exists a semi-universal family* $(\tilde{\mathfrak{X}}, \tilde{\mathfrak{D}})$ *of logarithmic deformations of* $(\tilde{X}, \tilde{D})$. *Moreover, blowing down* $\tilde{\mathfrak{D}}$, *we obtain a flat deformation* $\mathfrak{D}$ *of* D.

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