Paper Communicated.

Algebraic Means.

By Rikitarō FUJISAWA.

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For several years, interrupted only by an illness two years ago which almost proved fatal, my thought has been constantly directed to getting insight into the axiomatic nature of the means or averages of any number of positive quantities. Hereby the word *quantity* is used as synonymous with *number*.

The present communication is fragmentary in substance and somewhat vague in many respects. I crave the indulgence of the reader for this and many other shortcomings which it may not be necessary to enumerate in this place.

§. 1.

Gauss in his *Theoria Motus Corporum Cœlestium* (Werke Bd. VII, p. 232) takes it as an axiom, that, if any quantity has been determined by direct observations, the arithmetic mean of all the observed values is its most probable value, and, even if this be not strictly true, it is the nearest approach to the most probable value, so that we may safely accept it as such. Encke's so-called proof that the arithmetic mean is the most probable value for any number of observations (*Berliner Astronomisches Jahrbuch* for 1834, pp. 260–262), though consecrated by lapse of time, is not free of serious objections, and the foundation on which it rests, may ultimately be traced to the principle called *the equal distribution of ignorance* by Boole and *the want of sufficient reason* by De Morgan, for which I venture to suggest the name *no reason for preference*, and which was ably refuted by Johannes von Kries and his followers.

Whenever men of exact sciences hear of a *mean* or an *average*, they are liable to think at once of the very special case, in which we strive to arrive at the nearest approach to the single true value, the existence of which is tacitly assumed and which can never be found. In such a case, the given quantities which are usually the result of measurements or observations and consequently also the mean differ from one another but slightly. Now, in order to discuss the mean of any number of given quantities from a more general standpoint of view, it is first of all necessary to get rid of such an unwarrantable preconception. A mean of any number of positive quantities is a characteristic representative in a certain aspect of the aggregate or collectivity of these quantities. It is analogous to the representative in one capacity or another of a group of persons, among whom there may exist wide differences of opinion and inclination.

§. 2.

Let there be n positive quantities $x_1, x_2, \ldots\ldots, x_n$. Unless they are all equal, there will be at least one which is the greatest and one which is the smallest among them. A *mean*, in the most general sense of the word, may be defined as follows: A mean of any number of positive quantities is a value intermediate between the greatest and the least, which represents a certain characteristic of the collectivity of these quantities, and which, consequently, depends upon every one of them. It may be defined by algebraic operations upon given quantities as in the cases of the arithmetic mean and the geometric mean, or by transcendental operations as in the cases of the median and Gauss' arithmetico-geometric mean, or, again, by a series of operations which it would hardly be possible to incorporate in an analytical formula and which can only be most conveniently enunciated verbally as in the case of the mode often made use of in statistical investigations. Indeed, in view of the potentiality which most likely lies beyond the scope of our thought at the present primitive stage, it seems highly desirable to avoid as far as possible the use of functional symbols in the general investigations of means.

§. 3.

The axiomatic foundation of the theory of the arithmetic mean has been recently discussed by several authors such as G. Schiaparelli, U. Broggi, R. Schimmack and others. However, the result arrived at thus far seems to me, is not free of objections of giving undue prominence to formalism at the expense of the fundamental ideas underlying, and lacks that simplicity and perfection, which, according to Hilbert,* were so fittingly described by an eminent French mathematician of past days, who said: "une theorie mathématique ne doit être regardée comme parfaite que si elle a été renduc tellement claire qu'on paisse la faire comprendre au premier individu rencontré dan la rue."

Merely as examples of such properties or characteristics as are common to all thinkable kinds of means, we may mention the following:

a) If the given positive quantities are all equal, then their mean coincides with this equal value.

b) The mean of any number of the given positive quantities is independent of the order in which they may be arranged.

Now, suppose we know all the non-interdependent properties which are common to all kinds of means. By adding to these such specific properties of a particular kind of means, e. g. the arithmetic mean, we shall be lead to arrive at the particular mean under consideration. Only, through some such procedure, we may hope to attain to that degree of simplicity and perfection spoken of in the above in laying the axiomatic foundation

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* Compte Rendu de Deuxieme Congrès International des Mathematiciens. p. 59, Paris, 1902.

of the theory of the arithmetic mean or any other particular kind of means.

In order to find a system of the non-interdependent properties which are common to all kinds of means, it seems to be, first of all, necessary to widen the scope of our mental vision into the true nature of the most general conception of means. For this purpose, it was thought highly desirable, if not necessary, to study as many different kinds of means as we can possibly think of. It was in this connection that I was lead to consider a *scale* of means, analogous to the thermometric scale in the case of temperature, by which we may readily judge of the comparative *greater or less* of a particular mean in reference to other kinds of means.

§. 4.

Power Means

As before, let x_1, x_2, ..., x_n denote the given positive quantities. Divide the sum of the k^{th} powers of these quantities by n and let the k^{th} root of the quotient be denoted by P_k. Since

$$P_2^2-P_1^2=\frac{1}{n^2}\Sigma(x_1-x_2)^2,$$

it follows $P_2>P_1$. Again

$$P_{k-1}^{k-1}P_{k+1}^{k+1}-P_k^{2k}=\frac{1}{n^{2k}}\Sigma x_1^{k-1}x_2^{k-1}(x_1-x_2)^2\,;$$

whence follows

$$P_{k+1}^{k+1}>\left\{\frac{P_k}{P_{k-1}}\right\}^{k-1}P_k^{k+1}$$

Thus, provided $P_k>P_{k-1}$, $P_{k+1}>P_k$, and, since $P_2>P_1$, it follows

$$\ldots\ldots\ldots>P_k>P_{k-1}>\ldots\ldots\ldots>P_2>P_1$$

Observe that P_∞ is equal to the greatest of the given quantities and P_1 is the arithmetic mean.

Hitherto k was supposed to be positive. Now we may retain the same notation for negative indices, and it is easy to see that

$$P_{-1} > P_{-2} > \cdots\cdots\cdots > P_{-k} > \cdots\cdots\cdots\cdots$$

Hereby observe that P_{-1} is the harmonic mean which is always less than the geometric mean, and $P_{-\infty}$ is equal to the smallest of the given quantities.

For P_k, k denoting positive or negative integers, I propose the name of *power means*.

§. 5.

The Scale of Means.

For the interval between the arithmetic mean and the geometric mean, it was thought desirable to introduce more minute gradation by putting in means which lie intermediate between these two important means.

Let M_k, k denoting one of the integers 1, 2, ..., n, be defined by

$$M_k = \sqrt[k]{\frac{\Sigma x_1 x_2 \cdots x_k}{{}_nC_k}}$$

where Σ refers to all the combinations of the indices taken k at a time of 1, 2, ... n, and ${}_nC_k$ denotes the number of such combinations. For M_k, I propose the name of *algebraic means*. As will be shewn later,

$$M_1 > M_2 > \cdots\cdots\cdots > M_n.$$

Observe that M_1 in the arithmetic mean and M_n the geometric mean.

The sequence

$$P_\infty, \ldots\ldots, P_3, P_2, M_1, M_2, \ldots\ldots, M_n, P_{-1}, P_{-2}, \ldots\ldots, P_{-\infty}$$

is monotonic and continually decreasing. This I propose for *the scale of means.* It may be observed that we might insert between any two consecutive power means $n-1$ new means analogous to the algebraic means. However we shall leave the scale as it stands in the above.

§. 6.

Algebraic Means.

The proposition that

$$M_1 > M_2 > \dots\dots > M_n.$$

has already been proved by several authors, in some cases apparently without knowing that it was proved before. It is an example of the instances of frequent occurrence, where some mathematical truth is as often discovered as is forgotten or lost sight of.

The proof given by G. Darboux* is at once simple and elegant. Consider the equation.

$$x^n - {}_nC_1 M_1 x^{n-1}y + {}_nC_2 M_2^2 x^{n-2}y^2 - \dots\dots + (-1)^n M_n^n y^n = 0,$$

whose roots are all real and positive. Differentiating $k-1$ times with respect to x and $n-k-1$ times with respect to y, we get

$$M_{k-1}^{k-1} x^2 - 2M_k^k\, xy + M_{k+1}^{k+1} y^2 = 0.$$

Of this equation, we know that its two roots are both real. Hereafter, we shall always denote the discrimant of this equation by D; thus

$$D = M_k^{2k} - M_{k-1}^{k-1} M_{k+1}^{k+1}.$$

Since D is positive,

$$M_k^{k+1} > \left(\frac{M_{k-1}}{M_k}\right) M_{k+1}^{k+1}.$$

Since $M_1 > M_2$, as may easily be shewn, it follows from the above inequality that $M_k > M_{k+1}$.

§. 7.

The difference between the arithmetic and the geometric means expressed in the form of the sum of essentially positive terms was, so far as I am aware of, found by A. Hurwitz.** As

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* Bulletin des Sciences Mathématiques, Deuxieme Série Tome XXVI, p. 183, 1902.

** Ueber den Vergleich des Arithmetischen und des Geometrischen Mittels. Crelle's Journal Bd. 108, pp. 266–8, 1891.

the algebraic means seem to be destined to play an important rôle in the general theory of means, it was thought desirable to express D as a sum of essentially positive terms, and, moreover, it seemed to me not difficult to do so. It would hardly be necessary to add that, within reach of my search and inquiry, this was not hitherto done. This problem, together with some hints for its solution, was proposed to the students working in my seminary. The following is the solution obtained by R. Kurokawa, and, independently of him, by S. Kondo and others.

Let Σ denote the summation, whereby all the combinations of the indices are taken into account By actual calculation, we find:

$$M_1^2-M_0^0M_2^2=\frac{1}{n^2(n^2-1)}\Sigma(x_1-x_2)^2,$$

$$M_2^4-M_1^1M_3^3=\frac{1^2.2}{n^2(n-1)^2(n-2)}\{\Sigma x_1^2(x_2-x_3)^2+\Sigma x_1x_2(x_3-x_4)^2\},$$

$$M_3^6-M_2^2M_4^4=\frac{1^2.2^2.3}{n^2(n-1)^2(n-2)^2(n-3)}\times$$

$$\{\Sigma x_1^2x_2^2(x_3-x_4)^2+\Sigma x_1^2x_2x_3(x_4-x_5)^2+2\Sigma x_1x_2x_3x_4(x_5-x_6)^2\}.$$

Hereby M_0 may denote any quantity and is introduced solely for the sake of keeping the uniformity of the expressions on the left hand side.

In view of the above, let us assume generally

$$D=\frac{1}{k(k+1)_nC_{k\ n}C_{k+1}}\times$$

$$\left\{\sum_{r=0}^{k-1}A_r\left[\Sigma\, x_1^2x_2^2\ldots x_r^2x_{r+1}\ldots x_{2k-r-2}(x_{2k-r-1}-x_{2k-r})^2\right]\right\},$$

where A_r denote numerical coefficients. Replacing D by its expression in terms of the variables, we get

$$k(n-k)\{\Sigma x_1x_2\ldots x_k\}^2-(k+1)(n-k+1)\{\Sigma x_1x_2\ldots x_{k-1}\}\{\Sigma x_1x_2\ldots x_{k+1}\}$$

$$=\sum_{r=0}^{k-1}A_r\left[\Sigma x_1^2x_2^2\ldots x_r^2x_{r+1}\ldots x_{2k-r-2}(x_{2k-r-1}-x_{2k-r})^2\right].$$

Comparing the coefficients of $x_1^2x_2^2\ldots\ldots x_r^2x_{r+1}\ldots\ldots x_{2k-r}$ on the two sides, we obtain

$$k(n-k)\,{}_{2k-2r}C_{k-r}-(k+1)(n-k+1)\,{}_{2k-2r}C_{k-r-1}$$
$$=r(n-2k+r)A_{r-1}-(2k-2r)(2k-2r-1)A_r$$

Transposing,

$$(2k-2r)(2k-2r-1)A_r-r(n-2k+r)A_{r-1}$$
$$=\{k(k+1)-r(n+1)\}\frac{(2k-2r)!}{(k-r)!\,(k-r+1)!},$$
$$r=0,1,2,\ldots,k-1.$$

Observing that A_{-1} is identically zero, we readily find

$$A_0=\frac{(2k-2)!}{k!\,(k-1)!},\qquad A_1=\frac{(2k-4)!}{(k-1)!\,(k-2)!}.$$

In view of the above, we are lead to assume

$$A_r=\frac{(2k-2r-2)!}{(k-r)!\,(k-r-1)!}.$$

That this is true may be proved by mathematical induction; or, otherwise, we may proceed as follows. Putting

$$\frac{(k-r)!\,(k-r-1)!}{(2k-2r-2)!}=\psi_r,$$

the equation involving A_r and A_{r-1} becomes

$$(k-r)(k-r+1)\psi_r-r(n+2k+r)\psi_{r-1}=k(k+1)-r(n+1).$$

That is $\qquad (k-r)(k-r+1)\{\psi_r-1\}=r(n+2k+r)\{\psi_{r-1}-1\}.$

Obviously $\psi_1-1=0$, and as the factors $k-r$ and $k-r+1$ cannot vanish for values of r not exceeding $k-1$, it follows $\psi_r-1=0$. Thus, finally, we obtain

$$D=\frac{1}{k(k+1)\,{}_nC_k\,{}_nC_{k+1}}\times$$
$$\left\{\sum_{r=0}^{k-1}\frac{(2k-2r-2)!}{(k-r)!\,(k-r-1)!}\left[\Sigma x_1^2x_2^2\ldots x_r^2x_{r+1}\ldots x_{2k-r-2}(x_{2k-2r-1}-x_{2k-r})^2\right]\right\}$$

The variables being all positive, the summands on the right hand side are manifestly all positive. In passing by, we may observe that the coefficient before the bracket may also be written

$$\frac{1}{k(n-k)\,{}_nC^2_k},$$

and, furthermore, that the expression within the square bracket may be written

$$\Sigma(x_1-x_2)^2x_3^2x_4^2\ldots x_{r+2}^2\,x_{r+3}\ldots x_{2k-r}.$$

§. 8.

The result obtained in the last section, so far as it concerns the application to the theory of algebraic means, leaves but little to be desired. However, putting aside just for a while the question of algebraic means, and, considering D merely as an algebraic rational integral expression of n variables, a slight meditation will show that it is essentially positive, that is to say, independently of the sign of the variables. It is homogenous of the degree $2k$ in all the variables and of the second degree in each of them, and can only vanish when the variables become all equal. Such a consideration has lead to the conjecture of the existence of the form, into which D may be transformed and which clearly shows that it is positive, irrespective of whether these variables be positive or negative. This hint was given to the members of my mathematical seminary, several of whom succeeded in arriving at the anticipated result. The following is due to R. Kurokawa.

Consider

$$\sum_{r=0}^{k-1}\frac{(2k-2r-2)!}{(k-r)!\,(k-r-1)!}\Big[\Sigma(x_1-x_2)^2x_3^2x_4^2\ldots x_{r+2}^2x_{r+3}\ldots x_{2k-r}\Big].$$

Herein, the coefficient of $(x_1-x_2)^2$ is

$$\sum_{r=0}^{k-1}\frac{(2k-2r-2)!}{(k-r)!\,(k-r-1)!}\Big[\Sigma x_3^2x_4^2\ldots x_{r+2}^2x_{r+3}\ldots x_{2k-r}\Big]$$

(Indices 3, 4, ..., n)

On the other hand, the coefficient of $x_3^2x_4^2\ldots\ldots x_{r+2}^2x_{r+3}\ldots\ldots x_{2k-r}$ in the expression

$$\{\Sigma x_3x_4\ldots x_{k+1}\}^2+\frac{1}{{}_{k-1}C_1}\Sigma x_3^2\{\Sigma x_4\ldots x_{k+1}\}^2+\frac{1}{{}_{k-1}C_2}\Sigma x_3^2x_4^2\{\Sigma x_5\ldots x_{k+1}\}^2$$
$$+\ldots+\frac{1}{{}_{k-1}C_r}\Sigma x_3^2\ldots x_{r+2}^2\{\Sigma x_{r+3}\ldots x_{k+1}\}^2+\ldots\ldots+\Sigma x_3^2x_4^2\ldots x_{k+1}^2$$

(Indices 3, 4, ..., n)

is easily seen to be

$${}_{2k-2r-2}C_{k-r-1}\left\{1+\frac{{}_rC_1}{{}_{k-1}C_1}+\frac{{}_rC_2}{{}_{k-1}C_2}+\ldots\ldots\ldots+\frac{{}_rC_r}{{}_{k-1}C_r}\right\}.$$

On reduction, we find this to be equal to

$$k.\frac{(2k-2r-2)!}{(k-r)!\ (k-r-1)!}.$$

Since the above expression contains only the terms of the form $x_3^2x_4^2\ldots\ldots x_{r+2}^2\ x_{r+3}\ldots\ldots x_{2k-r}$, it follows that this expression divided by k is equal to the coefficient of $(x_1-x_2)^2$ in the expression further above. Thus we get

$$D=\frac{1}{k^2(k+1)_nC_k\ {}_nC_{k+1}}\Sigma(x_1-x_2)^2\sum_{r=0}^{k-1}\frac{1}{{}_{k-1}C_r}\ \Sigma x_3^2x_4^2\ldots x_{r+2}^2\{\Sigma x_{r+3}\ldots x_{k+1}\}^2$$

§. 9.

The Difference between the n^{th} Powers of the Arithmetic and the Geometric Means expressed as a Sum of Essentially Positive Terms.

Let us consider

$$M_1^{\ 1}\,M_{k-1}^{k-1}-M_k^{\ k}=\frac{\Sigma x_1}{n}\cdot\frac{\Sigma x_1x_2\cdots x_{k-1}}{{}_nC_{k-1}}-\frac{\Sigma x_1x_2\cdots x_k}{{}_nC_k}.$$

Without much difficulty, we find

$$M_1M_{k-1}^{k-1}-M_k^k=\frac{1}{nk.{}_nC_k}\ \Sigma(x_1-x_2)^2x_3x_4\cdots x_k.$$

Instead of k, writing $k-1$, $k-2$, ..., 2 in succession,

$$M_1\,M_{k-2}^{k-2}-M_{k-1}^{k-1}=\frac{1}{n(k-1).{}_nC_{k-1}}\ \Sigma(x_1-x_2)^2x_3x_4\cdots x_{k-1},$$

....,

$$M_1\,M_1^{\ 1}-M_2^{\ 2}=\frac{1}{n.\,2.\,{}_nC_2}\ \Sigma(x_1-x_2)^2.$$

Multiplying these equations by M_1^0, M_1^1, ..., M_1^{k-2} successively, and, adding, we obtain

$$M_1^{\ k}-M_k^{\ k}=\frac{1}{n}\sum_{r=0}^{k-2}\frac{M^r}{(k-r).{}_nC_{k-r}}\ \Sigma(x_1-x_2)^2x_3x_4\cdots x_{k-r},$$

where, on the right hand side, we have written M instead of M_1.

For $k=n$, the above equation becomes

$$\left\{\frac{x_1+x_2+\cdots\cdots+x_n}{n}\right\}-x_1x_2\cdots x_n$$
$$=\frac{1}{n}\sum_{r=0}^{n-2}\frac{M^r}{(n-r)\cdot {}_nC_{n-r}}\Sigma(x_1-x_2)^2x_3x_4\cdots\cdots x_{n-r}.$$

This seems to me to be preferable to the result obtained by Hurwitz in the paper already referred to in section 7.

In passing by, we may note that, eliminately M_1 between the equations

$$M_1\,M_{k-1}^{k-1}-M_k^k=\frac{1}{n.\,k.\,{}_nC_k}\Sigma(x_1-x_2)^2x_3x_4\cdots\cdots x_k,$$
$$M_1\,M_k^k-M_{k+1}^{k+1}\frac{1}{n(k+1)\cdot{}_nC_{k+1}}\Sigma(x_1-x_2)^2x_3x_4\cdots\cdots x_{k+1},$$

we shall obtain an expression for $M_k^{2k}-M_{k-1}^{k-1}\,M_{k+1}^{k+1}$ i.e. D, which, on reduction, will be found to agree with the foregoing result.

§. 10.

Algebraic Means of Different Orders. The Fundamental Mean.

Let us start from n positive quantities x_1, x_2, ..., x_n. Without any loss of generality, we may assume these quantities to have been arranged in the order of magnitude, so that x_1 is the greatest and x_n the smallest of them. Form the algebraic means of these quantities, and denote them, not by M_1, M_2, ..., M_n as was hitherto done, but by x_1' x_2' ......, x_n'. Again form the algebraie means of x_1', x_2', ..., x_n' and denote them by x_1'', x_2'', ..., x_n''. Further form the algebraic means of x_1'', x_2'', ..., x_n'' and denote them by x_1''', x_2''', ..., x_n''', and so on. I propose to call these successive aggregates each consisting of n quantities the algebraic means of the *first* order, of the *second* order, and so forth.

Now we know that

$$x_1' > x_2' > \dots\dots\dots\dots\dots\dots > x_n',$$

$$x_1'' > x_2'' > \dots\dots\dots\dots\dots\dots > x_n'',$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots,$$

$$x_1^{(m)} > x_2^{(m)} > \dots\dots\dots\dots > x_n^{(m)}.$$

On the other hand,

$$x_1 > x_1' > x_1'' > \dots\dots\dots, \quad \dots\dots\dots > x_n'' > x_n' > x_n.$$

In $x_1' - x_n'$ *i. e.* $\frac{x_1+x_2+\dots+x_n}{n} - \sqrt[n]{x_1 x_2 \dots x_n}$, replace each of x_2, x_3, ..., x_{n-1} in the minuend by x_1 and each of x_1, x_2, ..., x_{n-1} in the subtrahend by x_n.

Thus $$x_1' - x_n' < \frac{n-1}{n}(x_1 - x_n);$$

hence $$x_1^{(m)} - x_n^{(m)} < \left(\frac{n-1}{n}\right)^m (x_1 - x_n).$$

Provided n be finite, the left hand side of the above inequality converges toward zero when m becomes greater and greater and ultimately infinitely great. Thus it is proved that every one of the algebraic means of the mth order converges toward one and the same definite limit, when m becomes infinitely great. I propose to call this limiting value the *fundamental mean* of the given quantities x_1, x_2, ..., x_n.

In the particular case $n=2$, the fundamental mean coincides with Gauss' arithmetico-geometric mean λ of x_1 and x_2 given by

$$\frac{1}{\lambda} = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{x_1^2 \cos^2\theta + x_2^2 \sin^2\theta}}.$$
