17. Approximation of an Irrational Number by Rational Numbers.

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1. Let ω be a positive irrational number, $[a_0, a_1, a_2, \cdots\cdots]$ its expansion into the simple continued fraction, and

$$P_n/Q_n = [a_0, a_1, a_2, \cdots\cdots, a_n],$$
$$P_{\lambda, n}/Q_{\lambda, n} = [a_{\lambda+1}, \cdots\cdots, a_n].$$

Prof. FUJIWARA[1] proved that the minimum of

$$Q_n^2\left|\omega - \frac{P_n}{Q_n}\right|, \quad Q_m^2\left|\omega - \frac{P_m}{Q_m}\right| \quad \text{and} \quad Q_l^2\left|\omega - \frac{P_l}{Q_l}\right|$$

is less than

$$\frac{1}{\sqrt{9 - \dfrac{4}{Q_{n,l}^2}}},$$

when $Q_{n,m}$, $Q_{m,l}$ and $Q_{n,l}$ satisfy MARKOFF's equation

$$x^2 + y^2 + z^2 = 3\,xyz$$

and $m-n$, $l-m$ are odd.

By his suggestion, I have determined the numbers m, l for any MARKOFF's period.

2. Adopting MARKOFF's notations[2], let $Q\,\{a, a_1, a_2, \cdots\cdots, a_k, 2\}$ and $\mathfrak{R}\ (2, a, a, \cdots\cdots\lambda, \lambda, 2)$ be MARKOFF's number and the period of the continued fraction corresponding to the period $\{a, a_1, a_2, \cdots\cdots a_k, 2\}$ respectively. Then from the relation

$$\{a+1, a_1, \cdots\cdots, a_k, 2\} = \{a, a_1, a_2, \cdots\cdots, a_k, 2\} + \{a_1 - 1, a_2, \cdots\cdots a_k, 2\}\ (k = \text{odd}),$$
$$\text{or} \qquad = \{a_1 - 1, a_2, \cdots, a_k, 2\} + \{a, a_1, a_2, \cdots\cdots a_k, 2\}\ (k = \text{even}),$$

(1) These Proceedings 2, 1–3.

(2) A. MARKOFF, Sur les formes quadratiques binaires, Math. Ann., 17 (1880), or Bachmann, Die Arithmetik der quadratischen Formen II, 106–129.

we have, as the period of the continued fraction corresponding to the period

$$\{a+1, a_1, \cdots\cdots, a_k, 2\},$$

$$\left.\begin{array}{l}\mathfrak{K}(2, 1, 1, a, a, \cdots\cdots, \quad a, a, 1, 1, 2, 2, a, a, \cdots \\ \qquad \cdots, a, a, 2, 2, 1, 1, a, a, \cdots\cdots, a, a, 2)\end{array}\right\} (k=\text{odd}),$$

or $$\left.\begin{array}{l}\mathfrak{K}(2, 1, 1, a, a, \cdots\cdots, \quad a, a, 2, 2, 1, 1, a, a, \cdots \\ \qquad \cdots, a, a, 1, 1, 2, 2, a, a, \cdots\cdots, a, a, 2)\end{array}\right\} (k=\text{even}),$$

where the number of the elements are even.

Now take the case k = odd, and put

$$P_1'/Q_1' = [2, 1, 1, a, a, \cdots\cdots, a, a, 1, 1, 2, 2, a, a, \cdots \cdots, a, a, 2],$$
$$P_2'/Q_2' = [2, 1, 1, a, a, \cdots\cdots, a, a, 2],$$
$$P_3'/Q_3' = [2, 1, 1, a, a, \cdots\cdots, \; a, a, 1, 1, 2, 2, a, a, \cdots$$
$$\cdots, \; a, a, 2, 2, 1, 1, a, a, \cdots \cdots, \; a, a, 2],$$

then

$$Q_1' = Q\{a, a_1, \cdots\cdots\cdots\cdots, a_k, 2\} = K(1\,1\,a\,a \cdots$$
$$\cdots a\,a\,1\,1\,2\,2 \;\; a\,a \cdots\cdots \; a\,a\,2)^{(1)}$$
$$Q_2' = Q\{a_1-1, \; a_2, \cdots\cdots, a_k, 2\} = K(1\,1\,a\,a \cdots\cdots \; a\,a\,2),$$
$$Q_3' = Q\{\, a+1, \; a_1, \cdots\cdots, a_k, 2\} = K(1\,1\,a\,a \cdots$$
$$\cdots a\,a\,1\,1\,2\,2\,a\,a \cdots\cdots \; a\,a\,2\,2\,1\,1\,a\,a \cdots\cdots \; a\,a\,2).$$

Again put

$$P_1''/Q_1'' = [2, 1, 1, a, a, \cdots\cdots, a, a, 1, 1],$$
$$P_2''/Q_2'' = [2, 2, a, a, \cdots\cdots, a, a, \; 2, 2, 1, 1, a, a, \cdots\cdots, a, a, 2],$$
$$P_3''/Q_3'' = P_3'/Q_3',$$

in which the number of the elements are odd, except the last, then

$$Q_1'' = K(1\,1\,a\,a \cdots\cdots \; a\,a\,1\,1) = K(1\,1\,a\,a \cdots\cdots \; a\,a\,2) = Q_2',$$
$$Q_2'' = K(2\,a\,a \cdots\cdots \; a\,a\,2\,2\,1\,1\,a\,a \cdots\cdots \; a\,a\,2)$$
$$= K(1\,1\,a\,a \cdots\cdots \; a\,a\,2\,2\,1\,1\,a\,a \cdots\cdots \; a\,a\,2) = Q'_1.$$

As Markoff already shew[(2)],

$$Q_1'^2 + Q_2'^2 + Q_3'^2 = 3\,Q_1'\,Q_2'\,Q_3',$$

so we have

$$Q_1''^2 + Q_2''^2 + Q_3''^2 = 3\,Q_1''\,Q_2''\,Q_3''.$$

For the case k = even, we take

(1) Using $K(b_0\, b_1 \cdots\cdots b_n)$ for Muir's symbol $K\binom{1\;\,1 \cdots\cdots 1}{b_0\; b_1 \cdots\cdots b_n}$.

(2) Maarkoff, Loc. cit.

$$Q_1'' = K(1\,1\,a\,a\cdots\cdots a\,a\,2\,2\,1\,1\,a\,a\cdots\cdots a\,a\,1\,1),$$
$$Q_2'' = K(2\,a\,a\cdots\cdots a\,a\,2),$$
$$Q_3'' = K(1\,1\,a\,a\cdots\cdots a\,a\,2\,2\,1\,1\,a\,a\cdots\cdots a\,a\,1\,1\,2\,2\,a\,a\cdots\cdots a\,a\,2) = Q_3',$$

then again we get

$$Q_1''^2 + Q_2''^2 + Q_3''^2 = 3Q_1''\,Q_2''\,Q_3''.$$

In the trivial cases $\mathfrak{K}(1)$, $\mathfrak{K}(2)$ and $\mathfrak{K}(2,1,1,\cdots\cdots,1,1,2)$, it is sufficient to take

$$Q_1'' = K(0) = 1, \quad Q_2'' = K(0) = 1, \quad Q_3'' = K(1) = 1;$$
$$Q_1'' = K(0) = 1, \quad Q_2'' = K(0) = 1, \quad Q_3'' = K(2) = 2;$$

and $Q_1'' = K(1\,1\cdots\cdots 1\,1)$, $Q_2'' = K(0) = 1$, $Q_3'' = K(1\,1\cdots\cdots 1\,1\,2)$

respectively.

Therefore, if

$$\omega = [a_0, a_1, a_2, \cdots\cdots\cdots]$$

and

$$(a_{n+1}, a_{n+2}, \cdots\cdots\cdots, a_l) \qquad (l-n = \text{even})$$

be one of MARKOFF's periods, then we can determine m such that $l-m$, $m-n$ are odd and

$$Q_{n,m}^2 + Q_{n,l}^2 + Q_{m,l}^2 = 3Q_{n,m}\;Q_{n,l}\;Q_{m,l},$$

consequently the minimum of

$$Q_n^2\left|\omega - \frac{P_n}{Q_n}\right|, \quad Q_m^2\left|\omega - \frac{P_m}{Q_m}\right|, \quad Q_l^2\left|\omega - \frac{P_l}{Q_l}\right|$$

is less than

$$\frac{1}{\sqrt{9 - \dfrac{4}{Q_{n,l}^2}}}.$$

Special cases where $(a_{n+1}, \cdots\cdots, a_l) = (2,1,1,2)$, $(2,1,1,1,1,2)$, $(2,1,1,2,2,2)$ are treated in Prof. FUJIWARA's Note.