37. *On Some Properties of Orthogonal Functions.*

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Let G be a simply connected domain bounded by an analytic curve C of length l on a *Gaussian* plane; and consider a set of functions $V_\nu(z)$, $(\nu = 0, 1, 2, \dots)$ which are regular and analytic for all values of z in C and form a complete system of normalized orthogonal functions on C, that is,

$$\frac{1}{l}\int_C V_\nu(\xi)\overline{V_\mu(\xi)}ds \begin{cases} = 0 & \text{for } \nu \neq \mu, \\ = 1 & \text{for } \nu = \mu. \end{cases}$$

Then the series

$$\sum_{\nu=0}^{\infty} V_\nu(z)\overline{V_\nu(a)}, \ (a \text{ in } C)$$

is convergent absolutely and uniformly for all values of z in C and represents a *definite* function $K(z, a)$, which is regular and analytic in C and is defined only by the curve C; and a function $f(z)$, which is regular and analytic in C and is squarely integrable on C can be expressed by the following formula [1]:

$$f(z) = \frac{1}{l}\int_C f(\xi)K(z, \xi)ds, \ (z \text{ in } C).$$

By making use of this formula we can prove the following theorem:

Theorem 1. *If a set of functions $\{f(z)\}$ have the properties:*

(i) *$f(z)$ is regular and analytic for all values of z in C,*

(ii) $f(a) = 0$, $(a$ in $C)$,

(iii) $\frac{1}{l}\int_C |f(\xi)|^2 ds \leq M, \ (M > 0),$

then among such functions the unique one which gives the maximum of $|f(x)|$, $(x$ in $C)$ is given by

1) S. TAKENAKA, On the orthogonal functions and a new formula of interpolation. This paper will appear in Japanese Journ. of Math. 3 (1926).

$$(1)\quad f(z) = \varepsilon' M^{\frac{1}{2}} \frac{K(z, a)K(a, x) - K(z, x)K(a, a)}{(K(a, a))^{\frac{1}{2}}\{K(x, x)K(a, a) - K(x, a)K(a, x)\}^{\frac{1}{2}}}\ ^{1)},$$

$(|\varepsilon'| = 1)$,

or

$$(2)\quad f(z) = \varepsilon'' M^{\frac{1}{2}} \chi(z, x) \frac{K(z, x)}{(K(x, x))^{\frac{1}{2}}}, \quad (|\varepsilon''| = 1)\ ^{2)},$$

where

$$\chi(z, a) = e^{-g(z, a) - ih(z, a)}$$

in which $g(z, a)$ *is Green's function of* G *whose pole is at the point* a *in* C *and* $h(z, a)$ *is a conjugate harmonic function of* $g(z, a)$.

The first formula may be proved by the theory of linear integral equations, and the second by the relations between the generalized geometrical and arithmetical means applied on generalized POISSON-JENSEN's formula recently given by F. and R. NEVANLINNA [3].

Now, comparing these two formulas, we have

$$(3)\quad \chi(z, a) = \varepsilon \left\{1 - \frac{K(a, x)K(z, a)}{K(a, a)K(z, x)}\right\}\left\{1 - \frac{|K(x, a)|^2}{K(a, a)K(x, x)}\right\}^{-\frac{1}{2}},$$
$$(|\varepsilon| = 1,\ a \neq x).$$

It is remarkable that the absolute value of the right hand side of (3) is independent of χ.

Equating the absolute value of both sides of (3) and writing $z = x$, we have

$$e^{-g(z,a)} = \left\{1 - \frac{|K(z, a)|^2}{K(a, a)K(z, z)}\right\}^{\frac{1}{2}}.$$

Thus *Green*'s function may be expressed by the following formula:

$$g(z, a) = -\frac{1}{2} \log\left\{1 - \frac{|K(z, a)|^{\frac{1}{2}}}{K(a, a)K(z, z)}\right\}.$$

Therefore a theorem which I have proved in the paper cited [2] may be stated as follows:

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1) S. TAKENAKA, General mean modulus of analytic functions, Theorem 2. This will appear in the Tôhoku Math. Journal.

2) S. TAKENAKA, General mean modulus of meromorphic functions, Theorem 4. This will appear in the Japanese Journal of Math. 3 (1926).

3) F. and R. NEVANLINNA, Ueber die Eigenschaften analytischer Funktionen in der Umgebung einer singulären Stelle oder Linie, Acta Soc. Fennicae, 50 (1922) Or see F. NEVANLINNA, Ueber die Beziehungen zwischen dem Anwachsen einer analytischen Funktion etc., Conférences faites au cinquièmes Congrès des Mathématiciens Scandinaves, Helsingfors (1923), 292.

Theorem 2. *Let $f(z)$ be regular and analytic for all values of z in C and have zero points a_ν, ($\nu = 1, 2, \dots, n$) in C; and suppose that $|f(\xi)|^p$ is integrable on C as well as $\log|f(z)|$.*

Then the following inequality holds for all values of z in C:

$$(4)\quad |f(z)| \leq \prod_{\nu=1}^{n}\left\{1-\frac{|K(z, a_\nu)|^2}{K(a_\nu, a_\nu)K(z, z)}\right\}^{\frac{1}{2}}\left\{K(z, z)\frac{1}{l}\int_C |f(\xi)|^p ds\right\}^{\frac{1}{p}}$$

in which p is an arbitary real constant not zero, and the necessary and sufficient condition that the equality holds good in (4) at a point $z = x$ is that

$$f(z) = \lambda(K(z, x))^{\frac{2}{p}}\prod_{\nu=1}^{n}\left\{1-\frac{K(a_\nu, x)K(z, a_\nu)}{K(a_\nu, a_\nu)K(z, x)}\right\}\left\{1-\frac{|K(x, a_\nu)|^2}{K(a_\nu, a_\nu)K(x, x)}\right\}^{-\frac{1}{2}},$$

in which λ is an arbitrary constant.