83. *On the Zero Points of a Bounded Analytic Function.*

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1. Let

(1) $$f(x) = 1 + a_1 x + \cdots + a_n x^n + \cdots\cdots$$

be regular for $|x| < 1$ and x_0 the zero point of $f(x)$ which has the smallest modulus. Without any further restriction on $f(x)$, there does not exist a positive constant ρ, such that $|x_0| \geqq \rho$ for all functions (1), as the following example shows: in fact $f(x) = 1 + nx$ has a root $x = -\dfrac{1}{n}$, whose modulus can be made very small by taking n sufficiently large. Now we impose on $f(x)$ the restriction:

(2) $$f(0) = 1, \quad \text{and} \quad |f(x)| < M \text{ for } |x| < 1$$ [1]

For brevity, we call such a function $f(x)$ *a function of class M,* and will prove the existence of a positive quantity ρ_n, which has the following properties:

1) Every function of class M has at most $n-1$ roots in the circle $|x| < \rho_n$.

2) Among the functions of class M, there exists a function which has just n roots in the circle $|x| \leqq \rho_n$.

As we shall see later, the number of roots of a function of class M in the circle $|x| \leqq \rho_n$ can not exceed n and when there are just n roots in the circle $|x| \leqq \rho_n$ all roots must lie on the circle $|x| = \rho_n$. We will call such a function of class M that has just n roots on the circle $|x| = \rho_n$ an *extremal* function.

Theorem I. Let ρ_n be the quantity defined above, then

$$\rho_n = \frac{1}{\sqrt[n]{M}},$$

the extremal functions are $f(x) = \dfrac{a_1 - x}{1 - \bar{a}_1 x} \cdot \dfrac{a_2 - x}{1 - \bar{a}_2 x} \cdot \cdots \dfrac{a_n - x}{1 - \bar{a}_n x} \cdot \dfrac{1}{a_1 \cdots a_n}$,

1) From $f(0) = 1$ and $|f(x)| < M$ we have $M > 1$.

where $|a_1| = |a_2| = \cdots = |a_n| = \frac{1}{\sqrt[n]{M}}$.

Proof. Let $f(x)$ be regular for $|x|<1$ and $|f(x)|<M$.

If $f(x)$ vanish at the points $x_1, x_2, \cdots x_n$ in the unit circle, then by Jensen's extension of Schwarz's theorem,[1] we have

$$|f(x)| \leqq M\left|\frac{x-x_1}{1-\bar{x}_1 x}\cdot\cdots\frac{x-x_n}{1-\bar{x}_n x}\right| \quad \text{for} \quad |x|<1, \tag{3}$$

where the equality holds only for

$$f(x) = e^{i\phi} M \prod_{k=1}^{n} \frac{x-x_k}{1-\bar{x}_k x}. \tag{4}$$

Putting $x=0$ in (3) and considering $f(0)=1$ we have

$$|x_1 \cdots x_n| \geqq \frac{1}{M}, \tag{5}$$

whence, supposing $|x_1| \leqq |x_2| \leqq \cdots \leqq |x_n|$, we get $|x_n|^n \geqq \frac{1}{M}$, or

$$|x_n| \geqq \frac{1}{\sqrt[n]{M}}. \tag{6}$$

Hence by the definition of ρ_n, we must have

$$\rho_n \geqq \frac{1}{\sqrt[n]{M}}. \tag{7}$$

Now we take $a_1, \cdots a_n$ so that $|a_1| = \cdots = |a_n| = \frac{1}{\sqrt[n]{M}}$, and form a function

$$f(x) = \frac{a_1-x}{1-\bar{a}_1 x} \cdots \frac{a_n-x}{1-\bar{a}_n x}\cdot\frac{1}{a_1\cdots a_n},$$

then $f(0) = 1$ and $|f(x)|<M$ for $|x|<1$, hence $f(x)$ belongs to the class M and has n roots on the circle $|x| = \frac{1}{\sqrt[n]{M}}$, so that by the definition of ρ_n, we must have

$$\rho_n \leqq \frac{1}{\sqrt[n]{M}}. \tag{8}$$

From (7) and (8) we get

$$\rho_n = \frac{1}{\sqrt[n]{M}}. \tag{9}$$

1) Jensen, Untersuchungen über eine Klasse fundamentaler Ungleichungen in der Theorie der analytischen Funktionen I, Klg. Dansk. Vidensk, Selk. skr. nat. og math. afd. (8) **23** (1916), 203–222.

Cf. G. Pólya und G. Szegö, Aufgaben und Lehrsätze aus der Analysis, I, 142. I am indebted to Mr. T. Shimizu for the suggestion of this inequality.

From (6) we see $|x_{n+1}| \geqq \frac{1}{\sqrt[n+1]{M}} > \frac{1}{\sqrt[n]{M}}$, hence in a circle $|x| \leqq \frac{1}{\sqrt[n]{M}}$ there exist at most n roots of $f(x)$.

Let $f(x)$ be an extremal function so that it has just n roots $x_1, \cdots x_n$ in the circle $|x| \leqq \frac{1}{\sqrt[n]{M}}$, then

(5) $$|x_1 \cdots x_n| \geqq \frac{1}{M}.$$

And since

(10) $$|x_1| \leqq \frac{1}{\sqrt[n]{M}}, \quad |x_2| \leqq \frac{1}{\sqrt[n]{M}}, \cdots \quad |x_n| \leqq \frac{1}{\sqrt[n]{M}},$$

we have

(11) $$|x_1 \cdots x_n| \leqq \frac{1}{M}.$$

From (5) and (11) and (10) we get

(12) $$|x_1 \cdots x_n| = \frac{1}{M}, \quad |x_1| = |x_2| = \cdots = |x_n| = \frac{1}{\sqrt[n]{M}}.$$

We see by (12) that the equality in the formula (3) holds for $x=0$, hence $f(x)$ must be of the form (4), and considering $f(0)=1$ we get,

$$f(x) = \frac{1}{x_1 \cdots x_n} \cdot \frac{x_1 - x}{1 - \bar{x}_1 x} \cdot \cdots \frac{x_n - x}{1 - \bar{x}_n x},$$

which completes the proof of the theorem.

2. As an application, consider an integral function of order ρ

$$f(x) = 1 + a_1 x + \cdots + a_n x^n + \cdots\cdots$$

where $|f(x)| < e^{r^{\rho'}}$ for $|x| \leqq r$, and ρ' is any quantity greater than ρ. Put $x = rz$ and $f(x) = \varphi(z)$, then we have

$$|\varphi(z)| < e^{r^{\rho'}} \quad \text{for} \quad |z| \leqq 1.$$

Hence if we denote the roots of $\varphi(z)$ in the unit circle by $z_1, \cdots z_n$ and suppose $|z_1| \leqq |z_2| \leqq \cdots \leqq |z_n|$, we have from (6) $|z_n| \geqq e^{-\frac{r^{\rho'}}{n}}$, or

(13) $$|x_n| \geqq re^{-\frac{r^{\rho'}}{n}},$$

where $x_1, \cdots x_n$ are the corresponding roots of $f(x)$ which lie in a circle $|x| \leqq r$. But if x_n lies outside the circle $|x| = r$, so that $|x_n| \geqq r$, then (13) evidently holds. Hence (13) holds for every r. Now the value of r

hich makes $re^{-\frac{r^{\rho'}}{n}}$ maximum is easily found to be

$$r_0=\left(\frac{n}{\rho'}\right)^{\frac{1}{\rho'}},$$

so that we have
$$|x_n|\geqq\left(\frac{n}{e\rho'}\right)^{\frac{1}{\rho'}}.$$

Hence

Theorem II. If $f(x)=1+a_1x+\cdots+a_nx^n+\cdots$ *is an integral function of order* ρ, *and* $x_1, x_2, \cdots x_n, \cdots$ *the roots of* $f(x)$ *in ascending order of absolute values, then*

$$|x_n|\geqq\left(\frac{n}{e\rho'}\right)^{\frac{1}{\rho'}},$$

where $\rho'>\rho$.

From this follows at once, for example, that $\sum_{n=1}^{\infty}\frac{1}{|x_n|^{\rho+\epsilon}}$ converges (Hadamard's theorem).