103. *On the Projective Differential Geometry of Plane Curves and One-Parameter Families of Conics.*

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1. Recently the projective differential geometry has been developed by many mathematicians, but they all considered points, lines or planes as elements. From this fact the projective differential invariants have lost, as we can readily see, their geometrical meaning not a little. In this paper I will develope the theory of projective differential geometry of plane curves and one-parameter families of conics by considering conics as elements.

2. Consider a one-parameter family of conics in the plane, and let its equation in homogeneous coordinates be

$$\sum a_{ik}(t)x_i x_k = 0.$$

We assume the determinant $|a_{ik}(t)|$ not identically zero, and normalize $a_{ik}(t)$ as follows:

$$\overset{*}{a}_{ik}(t) = |a_{ik}(t)|^{-\frac{1}{3}} a_{ik}(t)$$

which we consider as coordinates of the family of conics. In the following we assume a_{ik} as normalized. Apply a projective transformation

$$\overline{x}_i = a_{ij}\, x_j$$

to the family, then we have

$$|\overline{a}_{ik}(t)| = |a_{ik}(t)|\,|a_{ij}|^2.$$

so that we can assume without any loss of generality that

$$|a_{ij}| = \pm 1.$$

3. Next we introduce the following notation:

$$(l, m, n) = \frac{1}{2}\sum \begin{vmatrix} a_{11}^{(l)} & a_{12}^{(m)} & a_{13}^{(n)} \\ a_{21}^{(l)} & a_{22}^{(m)} & a_{23}^{(n)} \\ a_{31}^{(l)} & a_{32}^{(m)} & a_{33}^{(n)} \end{vmatrix}$$

where

$$a_{ik}^{(l)} = \frac{d^l}{dt^l} a_{ik}(t)$$

and Σ is extended over all permutations of (l, m, n). In this notation

$$|a_{ik}| = \frac{1}{3}(0,0,0) = 1$$

for normalized a_{ik}. $(1,1,0)$ is not in general identically zero, so that we introduce the following integral invariant as natural parameter

$$p = i \int \sqrt{(1,1,0)_t}\, dt,$$

which we shall call *the projective K-length* of the family. Accordingly we can easily show that

$$I_1 = (1,1,1),\ I_2 = (2,2,0),\ I_3 = (2,2,1),\ I_4 = (2,2,2),$$

$$I_5 = \sum A_{ik}^{(0,0)} A_{lm}^{(1,1)} A_{np}^{(2,2)} a_{mn} a'_{pi} a''_{kl}$$

are all projective differential invariants, where $A_{ik}^{(l,m)}$ is the coefficient of $a_{ik}^{(n)}$ in (l,m,n) and among these I_i and their derivatives a certain relation (A) must evidently exist.

4. Moreover we can prove the fundamental theorem :

Let us give

$$I_i = f_i(p) \qquad (i = 1,2,\cdots\cdots,5)$$

as functions of the projective K-length p, where f_1 is a five times continuously differentiable function and the others are all four times continuously differentiable functions, and among f_i and their derivatives the relation (A) *exists, then the family of conics with the five differential invariants I_i and the projective K-length p is uniquely determined, save as to the projective transformations.*

We can, therefore, consider $I_i = f_i(p)$ as the natural equations of the family of conics.

5. Since I_i are all algebraic invariants of three conics, which are represented by a_{ik}, a'_{ik}, a''_{ik} respectively, it is evident that I_i have some geometrical meaning.

6. If $(1,1,0)$ is identically zero, we introduce

$$p_2 = \int \sqrt[3]{(1,1,1)_t}\, dt$$

as natural parameter instead of p and we can see that the natural equations are $I_i = f_i(p_2)$ $(i = 2,3,4,5)$. When $(1,1,1)$ is also identically zero, we introduce

$$p_3 = \int \sqrt[4]{(2,2,0)_t}\,dt$$

as natural parameter and in this case the natural equations are $I_i = f_i(p_3)$ $(i = 3,4,5)$.

When (1,1,0), (1,1,1), (2,2,0) are all identically zero, we can easily prove that (2,2,1) and (2,2,2) are also identically zero. When (3,3,0) is also identically zero, the family consists of conics, having three or four pointic contact at a point with each other.

7. When (3,3,0) is not identically zero, the family consists of osculating conics of a plane curve. Accordingly it is sufficient in this case to consider such family of conics. First, we introduce

$$\sigma = \int \sqrt[9]{\frac{(3,3,3)_t}{3}}\,dt$$

as natural parameter, since

$$4(3,3,0)^3 + 3(3,3,3)^2 = 0.$$

Then we can prove that $(4,4,3) = f(\sigma)$ is the natural equation of the curve. We shall call (4,4,3) *the projective K-curvature.*

8. The dual theory of the above can be easily established by an analogous method. And this theory can be extended to three-dimensional space and to surfaces and other manifolds.