102. *On the Extension of a Theorem of Minkowski.*

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In these Proceedings Vol. 2, No. 3, I have communicated my researches about the order of $|ax-y+\beta|$, extending Klein's idea on geometrical interpretation of continued fractions. The following are the results which I have obtained since then concerning the same question.

Put
$$a = q_0 - \frac{\mu_1}{q_1 -}\ \frac{\mu_2}{q_2 -}\ \frac{\mu_3}{q_3 -}\ \cdots\cdots,$$
where $\mu_i = \pm 1$ and (q_i) are so taken that all
$$a_i = q_i - \frac{\mu_{i+1}}{q_{i+1}}\ -\ \frac{\mu_{i+2}}{q_{i+2}}\ -\ \frac{\mu_{i+3}}{q_{i+3}}\ -\ \cdots\cdots$$
will be greater than 2. Using these a_i, expand β in the following form:
$$\beta = p_0 + \frac{\mu_1}{a_1}p_1 + \frac{\mu_1\mu_2}{a_1a_2}p_2 + \frac{\mu_1\mu_2\mu_3}{a_1a_2a_3}p_3 + \cdots\cdots,$$
where (p_i) are so chosen that all
$$\beta_i = p_i + \frac{\mu_{i+1}}{a_{i+1}}p_{i+1} + \frac{\mu_{i+1}\mu_{i+2}}{a_{i+1}a_{i+2}}p_{i+2} + \frac{\mu_{i+1}\mu_{i+2}\mu_{i+3}}{a_{i+1}a_{i+2}a_{i+3}}p_{i+3} + \cdots\cdots$$
will be smaller than a_i. Evidently, if a and β are given, the sequence of numbers (q_i, p_i, μ_i) is determined uniquely. Then:

i. The necessary and sufficient condition for
$$\liminf |x(ax-y+\beta)| = \frac{1}{4}$$
is
$$q_i \to \infty \text{ and } \frac{p_i}{q_i} \to \frac{1}{2}.$$

ii. When q_i tends to infinity, $\liminf |x(ax-y+\beta)|$ can take any value smaller than $\frac{1}{4}$ when β is properly chosen. For example, if $q_i \to \infty$ and $\frac{p_i}{q_i} \to m < \frac{1}{2}$, then $\liminf |x(ax-y+\beta)| = m^2$.

iii. If $\liminf q_i = 2k$, where k is a positive integer, then

$$\liminf |x(\alpha x-y+\beta)| \leqq \frac{k}{4\sqrt{k^2+1}}.$$

The sign of equality occurs for and only for the form

$$\sqrt{k^2+1}\,x-y+\frac{\sqrt{k^2+1}-k+1}{2}$$

and its equivalent forms. All other forms satisfy the inequality

$$\liminf |x(\alpha x-y+\beta)| \leqq \frac{1}{4\sqrt{1+\frac{1}{k^2}+\frac{(2k+1)(4k^2+2k+1)}{4k^4(2k^2+2k+1)^2}}}.$$

iv. When $q_i = 2k$ and $\mu_{i+1} = 1$ for infinitely many i, we have

$$\liminf |x(\alpha x-y+\beta)| \leqq \frac{k(2k+1)}{4(2k^2+2k+1)}.$$

v. If $q_i = 2k+1$ for infinitely many i, then

$$\liminf |x(\alpha x-y+\beta)| \leqq \frac{1}{2\left(\sqrt{1+\frac{2}{k}}+\sqrt{1+\frac{1}{k^2}}\right)},$$

except the case $k = 1$, in which

$$\liminf |x(\alpha x-y+\beta)| \leqq \frac{1}{2\,\frac{\sqrt{221}}{5}}.$$