101. *On the System of Generalized Orthogonal Functions and its Relation to the Singular Integral Equations.*

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Let $\varphi(x, \lambda)$ be a real or complex continuous function of a real variable x in the interval $(0,\infty)$, and of a real parameter λ in a certain interval, such that $M\{\varphi(x, \lambda)\,\overline{\varphi}(x, \mu)\} = 0$ or 1, according as $\lambda \neq \mu$ or $\lambda = \mu$, where $\overline{\varphi}$ denotes the conjugate complex function of φ, and $M\{f(x)\}$ means after H. Bohr

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(x)\,dx.$$

We call the family of such functions the system of generalized orthogonal functions. We restrict ourselves in the following to the real function $\varphi(x, \lambda)$.

If $f(x)$ be a real continuous function for $x \geqq 0$, for which $M\{f^2(x)\}$ exists and >0, and further if it can be uniformly approximated by finite number of functions belonging to the system $\{\varphi(x, \lambda)\}$, i.e. for any $\varepsilon > 0$ there exist an integer N and two sequences of real numbers $(a_1, a_2, \cdots\cdots, a_N)$, $(\mu_1, \mu_2, \cdots\cdots, \mu_N)$, such that

$$\left| f(x) - \sum_1^N a_k \varphi(x, \mu_k) \right| < \varepsilon \text{ for all } x \geqq 0,$$

then Bohr's theory of the almost periodic functions may be extended to this case. For example, we can prove that for such a function $f(x)$ there corresponds an at most enumerable set of real numbers $(\lambda_1, \lambda_2, \cdots)$, for which $M\{f(x)\,\varphi(x, \lambda_k)\} = A_k \neq 0$, while for any other values of λ, $M\{f(x)\,\varphi(x, \lambda)\} = 0$, and further that $\sum A_k\,\varphi(x, \lambda_k)$ converges in means to $f(x)$, i.e.

$$\lim_{N\to\infty} M\left\{ f(x) - \sum_1^N A_k \varphi(x, \lambda_k) \right\} = 0,$$

or

$$M\left\{ f^2(x) \right\} = \sum_1^\infty A_k^2.$$

The aim of this note is to remark that some systems of generalized orthogonal functions have a close relation to the singular integral equations.

Let $L(u)=\frac{d}{dx}\left(p(x)\frac{du}{dx}\right)+q(x)\,u$, where $p(x)$, $p'(x)$, $q(x)$ are all continuous and $p(x)>0$ for $x>0$, and consider the boundary value problem for the differential equation of the second order $L(u)+\lambda u=0$, corresponding to the boundary condition of the following form:

(A). $u(x)$ and $u'(x)$ remain finite at both ends of the interval $(0, \infty)$, i.e. for $x\to 0$ and $x\to\infty$.

If Green's function $G(x, \xi)$ of the differential equation $L(u)=0$ satisfying the boundary condition (A) exists, then it results $G(\xi, \eta)=G(\eta, \xi)$, when it satisfies moreover the relation

$$p(x)\{G(x, \xi)\, G'_x(x, \eta)-G'_x(x, \xi)\, G(x, \eta)\}=0$$

for $x\to 0$, $x\to\infty$. If $u(x)$ be the solution of the differential equation $L(u)+\varphi(x)=0$, which is continuous with $u'(x)$, $u''(x)$ for $x>0$ and satisfies the condition

$$p(x)\{u(x)\, G'_x(x, \xi)-u'(x)\, G(x, \xi)\}=0$$

for $x\to 0$, $x\to\infty$, then we get

$$u(x)=\int_0^\infty \varphi(\xi)\, G(x, \xi)\, d\xi.$$

Consequently, if $u(x)=\varphi(x, \lambda)$ be the solution of $L(u)+\lambda u=0$ satisfying the above condition, then we arrive at the integral equation

$$(a)\quad \varphi(x, \lambda)=\lambda\int_0^\infty G(x, \xi)\, \varphi(\xi, \lambda)\, d\xi.$$

In this case, however, the kernel $G(x, \xi)$ is not regular, and the integral $\int_0^\infty \varphi^2(x, \lambda)\, dx$ does not converge in general; and the characteristic numbers λ are not discrete, but may be continuous. These facts were first remarked by Weyl and Picard.(1)

We can however prove from the form $L(u)+\lambda u=0$ the relation

$$M\{\varphi(x, \lambda)\varphi(x, \mu)\}=0 \text{ for } \lambda\neq\mu.$$

If therefore $M\{\varphi^2(x, \lambda)\}$ exists and $=c^2>0$, then we take $\varphi(x, \lambda)/c$ as the normalized characteristic function, and thus we see that the system of the normalized characteristic functions of the singular integral equation (a) forms a system of generalized orthogonal functions.

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(1) Weyl, Göttinger Dissertation, 1908; Math. Annalen, 68 (1910). Picard, Comptes Rendus, 1910; Ann. l'École norm., ser. III, 28 (1911). See also Hilb, Math. Annalen, 66 (1909).

If the integral $\int_0^\infty G(x, \xi)\ \varphi(x, \lambda)\ dx$ converges uniformly, then we can prove this fact directly from the integral equation (a), without making use of the differential equation $L(u)+\lambda u = 0$.

As an example we take $L(u) = u''-u$. This case was fully treated by Weyl and Hilb. Green's function and the characteristic functions are respectively

$$G(x, \xi) = e^{-\xi} \sinh x \quad \text{for } x \leqq \xi,$$
$$= e^{-x} \sinh \xi \quad \text{for } x > \xi,$$
$$\varphi(x, \lambda) = \sqrt{2} \sin (\sqrt{\lambda-1}\, x)\, ;\ (\lambda \geqq 1)$$

or

$$G(x, \xi) = e^{-\xi} \cosh x \quad \text{for } x \leqq \xi,$$
$$= e^{-x} \cosh \xi \quad \text{for } x > \xi,$$
$$\varphi(x, \lambda) = \sqrt{2} \cos (\sqrt{\lambda-1}\, x). \quad (\lambda \geqq 1).$$

As the second example we take $L(u) = u'' - \frac{1-4n^2}{4x^2}u$, $(n \geqq 1)$. Here Green's function is

$$G(x, \xi) = \frac{\sqrt{x\xi}}{2n}\left(\frac{x}{\xi}\right)^n \quad \text{for } x \leqq \xi,$$
$$= \frac{\sqrt{x\xi}}{2n}\left(\frac{\xi}{x}\right)^n \quad \text{for } x > \xi,$$

and the characteristic functions are

$$\varphi(x, \lambda) = \sqrt{\pi}\,(\sqrt{\lambda}\, x)^{\frac{1}{2}} J_n(\sqrt{\lambda}\, x), \ (\lambda \geqq 0).$$