15. *Asymmetric Vibrations of Finite Amplitudes.*

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In the former communication[1], asymmetric vibration of small amplitude was discussed, but the treatment of the problem does not show any difference when the amplitude is finite and presents many novel features, which have important bearings on many physical phenomena. Denoting the displacement by ξ, the equation of motion is

$$m\frac{d^2\xi}{dt^2} = -f\xi + g\xi^2 \tag{1}$$

If $g > 0$, then for finite displacement, the restitutive force becomes repulsive when ξ becomes so large that $f < g\xi$, so that the displacement may ultimately become infinitely great. It was formerly assumed that $g > 0$, but the case $g < 0$ can be treated exactly in the same manner.

The first integral of (1) being

$$\frac{m}{2}\left(\frac{d\xi}{dt}\right)^2 = a - \frac{f\xi^2}{2} + \frac{g}{3}\xi^3, \tag{2}$$

we obtain the equation

$$\frac{d\xi}{\sqrt{\xi^3 - \frac{3}{2}\frac{f}{g}\xi^2 + \frac{3a}{g}}} = \frac{d\xi}{\sqrt{(\xi-\alpha)(\xi-\beta)(\xi-\gamma)}} = \frac{d\xi}{\sqrt{R(\xi)}} = \sqrt{\frac{2g}{3m}}\,dt, \tag{3}$$

where α, β, γ are the roots of the cubic under the radical; they are all real when the condition $6ag^2 < f^3$ is satisfied. Suppose that $\alpha > \beta > \gamma$, and put $k^2 = \frac{\beta-\gamma}{\alpha-\gamma}$, $k'^2 = \frac{\alpha-\beta}{\alpha-\gamma}$, then

$$\frac{d\xi}{\sqrt{R(\xi)}} = \frac{2}{\sqrt{\alpha-\gamma}}\frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}} = \frac{2}{\sqrt{\alpha-\gamma}}du$$

$$= \frac{2i}{\sqrt{\alpha-\gamma}}\frac{dz}{\sqrt{(1-z^2)(1-k'^2z^2)}} = \frac{2i}{\sqrt{\alpha-\gamma}}du'$$

1) NAGAOKA: Proc. Imp. Acad. 3 (1927) 28.

Thus

$$u = \sqrt{\frac{g(\alpha-\gamma)}{6m}}\, t \text{ and } u' = \frac{1}{i}\sqrt{\frac{g(\alpha-\gamma)}{6m}}\, t \qquad (4)$$

by taking the origin of time properly. The displacement ξ can be expressed according to the following schedule, the limits of excursion being given in the second and third columns.

ξ	u real	u imaginary
$\alpha+\dfrac{\gamma-\alpha}{\mathrm{sn}^2(k')}$	$-\infty$ and γ	∞ and α
$\beta+\dfrac{\alpha-\beta}{\mathrm{cn}^2(k)}$	∞ and α	α and β
$\gamma+\dfrac{\beta-\gamma}{\mathrm{dn}^2(k')}$	α and β	γ and β
$\gamma+(\beta-\gamma)\mathrm{sn}^2(k)$	β and γ	γ and $-\infty$

(5)

Another procedure is to put $\xi = s+\dfrac{f}{2g}$; then

$$\frac{ds}{\sqrt{4s^3-g_2 s-g_3}} = \frac{ds}{\sqrt{S}} = \sqrt{\frac{g}{6m}}\, dt = -dv \qquad (6)$$

where $g_2 = \dfrac{3f^2}{g^2}$ and $g_3 = \dfrac{f^3}{g^3}-\dfrac{12a}{g}$, the roots of the cubic S being $e_1 > e_2 > e_3$, which we suppose to be all real, on the assumption that the discriminant of the cubic is greater than zero, or $6ag^2 < f^3$. The displacement is then given by the following values of s.

s	v real	v imaginary
$\wp v$	∞ and e_1	e_3 and $-\infty$
$\wp(v\pm\omega_1)$	∞ and e_1	e_1 and e_2
$\wp(v\pm\omega_2)$	e_2 and e_3	e_1 and e_2
$\wp(v\pm\omega_3)$	e_2 and e_3	e_3 and $-\infty$

(7)

Evidently (5) and (7) are not different from each other, although the notations are those of Jacobi and of Weierstrass resp. On expressing the Jacobi's functions involved in (5) by means of the formulae given in Fundamenta Nova[1]. we arrive at the same result as those deduced from $\wp$-function, of which the following can be employed with advantage.

$$\wp(v) = -\frac{\eta_1}{\omega_1}-\left(\frac{\pi}{2\omega_1}\right)^2 \operatorname{cosec}^2 \frac{\pi v}{2\omega_1} - \frac{2\pi^2}{\omega_1^2}\sum \frac{rq^{2r}}{1-q^{2r}} \cos \frac{\pi}{\omega_1} v$$

$$\wp(v\pm\omega_1) = -\frac{\eta_1}{\omega_1}-\left(\frac{\pi}{2\omega_1}\right)^2 \sec^2 \frac{\pi v}{2\omega_1} - \frac{2\pi^2}{\omega_1^2}\sum \frac{(-1)^r rq^{2r}}{1-q^{2r}} \cos \frac{r\pi}{\omega_1} v \qquad (8)$$

$$\wp(v\pm\omega_2) = -\frac{\eta_1}{\omega_1} - \frac{2\pi^2}{\omega_1^2}\sum \frac{rq^{r}}{1-q^{2r}} \cos \frac{r\pi}{\omega_1} v$$

$$\wp(v\pm\omega_3) = -\frac{\eta_1}{\omega_1} - \frac{2\pi^2}{\omega_1^2}\sum \frac{(-1)^r rq^{r}}{1-q^{2r}} \cos \frac{r\pi}{\omega_1} v.$$

The period of vibration is given by

$$T = 4\sqrt{\frac{6m}{g(a-\gamma)}}\,K = 4\sqrt{\frac{6m}{g}}\,\omega_1 \text{ for } g>0\text{; and}$$

$$T = 4\sqrt{\frac{6m}{g(a-\gamma)}}\,iK' = 4\sqrt{\frac{6m}{g}}\,\omega_3 \text{ for } g<0.$$

For vibratory motion between finite limits, the displacements are represented by cos. series as given by the last two formulae of (8). If we put $\tau = \frac{\pi}{2\omega_1} v$, the equation of motion (1) can be put in the form

$$\frac{d^2\xi}{d\tau^2} + \xi(p^2 + 2p_1 \cos 2\tau + 2p_2 \cos 4\tau + \cdots\cdots) = 0 \qquad (9)$$

This belongs to a particular case of the equation investigated by Hill in the lunar theory. In most cases q entering in (8) is a small quantity, so that $p_1, p_2, \cdots\cdots$ are small quantities. The case $p_2 = p_3 = \cdots\cdots = 0$ is the wellknown Gylden-Lindstedt equation in astronomy; thus the problem of forced vibration of an imperfectly elastic connection can be solved by borrowing the results of astronomers on the said equation, of which we have abundant literature.

The vibration between e_2 and e_3 was treated by Duffing in q-series, but the motion in other sections have not yet been studied. The excursion from $-\infty$ to e_3 and from e_1 to ∞ is of great interest, as the passage into infinity means break down of the elastic connection. The expressions for $\wp$ show that even during this transition there are vibratory motions of multiple frequencies. Though exact analogues could not be found, some inferences on the behaviour of over-strained elastic bodies before breaking can be made, as Hooke's law holds no more in such a state. Probably there are problems of geophysics whose solution can be approached from the result here obtained. Applications to many physical phenomena are reserved for future communications.

1) JACOBI: Gesammelte Werke, 1. (1881) pp. 168–169.
2) DUFFING: Erzwungene Schwingungen, (1918) 25, Braunschweig.