34. On a Property of Transcendental Integral Functions.

By Tatsujiro SHIMIZU.
Mathematical Institute, Tokyo Imperial University.

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Mr. Tsuji[1] proved that for a class of integral functions $f(z)$, for which $f(0) = a$, $f(z_i) = b$, $(i = 1,2,\cdots)$, where $a \neq b$, $a \neq 0, \neq 1$, and $b \neq 0, \neq 1$ and $|z_1| \leqq |z_2| \leqq \cdots \to \infty$, there exists an infinite number of concentric ring-regions $|z| < R_1$, $R_i < |z| < R_{i+1}$ $(i = 1,2,\cdots)$, R_i depending only on the class, in which all the functions of the class take at least once the value 1 or 0.

We will here prove the following allied

Theorem: *Consider a class of integral functions*

$$f(z) = c_0 + c_1 z + c_2 z^2 + \cdots + c_m z^m + \cdots\cdots, \tag{1}$$

for which $|c_m| \geqq \dfrac{l_0}{m!} > 0$ *for a certain value of* $m \geqq 1$, *and* $|f(z_i)| = l_i < M$ $(i = 1,2,\cdots)$, *where* l_i *are positive constants*[2] *and* $|z_1| \leqq |z_2| \leqq \cdots \to \infty$, *then there exists an infinite number of concentric ring-regions* $|z| < R_1$, $R_i < |z| < R_{i+1}$, $(i = 1,2,\cdots)$, R_i *depending only on the class, in which any function* (1) *takes at least once the value* 1 *or* 0, *and we can find an expression for an infinite number of radii* R_i *of the ring-regions* $R_i < |z| < R_{i+1}$.

Proof. Suppose, if possible, that a function (1) does not take the values 1 and 0 in the ring-region $0 \leqq R_0 < |z| < R$, $R = 2(r_i - R_0) + R_0$, where $|z_i| = r_i$, and therefore in the circle of radius $r_i - R_0$ with center at z_i, then by Landau's theorem[3] we have in $|z - z_i| < \dfrac{r_i - R_0}{2}$

$$|f(z)| < \Omega(M). \tag{2}$$

Now take $2q\left(q < \left[\dfrac{2\pi}{1 - R_0/r_i}\right] + 1\right)$ circles $C_{i,\pm h}$ $(h = 1,2,\cdots q)$ of radius

1) Proc. Imperial Academy, 2 (1926) 364–365.
2) In this case it is not necessary that $c_m \neq l_i$.
3) Götting. Nachr. (1910), 309.

$\frac{r_i - R_\jmath}{2}$ with centers on the circle $|z| = r_i$, so that they cover the whole circumference $|z| = r_i$, the center of $C_{i,\pm h}$ lying within $C_{i,\pm(h-1)}$, then by successive application of Landau's inequality to the circles of radii $r_i - R_\jmath$ about the same centers we have

$$|f(z)| < \Omega^{(q)}.(M)^{1)} \tag{3}$$

in the region covered by these circles $C_{i,\pm h}$. In this region and *a fortiori* in the circle $|z| \leqq r_i$ we have, as $q < [4\pi] + 1 = 13$, for $r_i > 2R_\jmath$,

$$|f(z)| < \Omega^{(13)}(M)^{2)}. \tag{4}$$

For all $r_i > 2R_\jmath$ we have from (4)

$$|c_m| \leqq \underset{|z|=r_i}{\text{Max}} |f(z)| \,/\, r_i^m \leqq \Omega^{(13)}(M) / r_i^m \tag{5}$$

$|c_m| \neq 0$ being given, for all r_i which satisfies the inequalities

$$r_i > 2R_0 \tag{6}$$

and
$$r_i^m > \frac{1}{|c_m|} \Omega^{(13)}(M), \tag{7}$$

the function (1) must assume at least once the value 1 or 0 in the ring-region $R_0 < |z| < 2r_i - R_\jmath$.

Hence for $R_\jmath = 0$ we obtain the circle $|z| < R_1 = 2r_{i_1}$, r_{i_1} satisfying (7), in which the function (1) takes the value 1 or 0. We can next take $R_2 = 2r_{i_2} - R_1$ as the outer radius of the ring-region $R_1 < |z| < R_2$, where

$$r_{i_2} > 2\, R_1, \tag{8}$$

and consequently by (7), (8) $r_{i_2}^m > \frac{1}{|c_m|} \Omega^{(13)}(M)$. Prcoeeding in this way we have in general

$$r_{i_p} > 2\, R_{p-1},\ R_p = 2\, r_{i_p} - R_{p-1} \qquad (p \geqq 1),$$

where
$$r_{i_1} > \sqrt[m]{\frac{1}{|c_m|} \Omega^{(13)}(M)}.$$

From this theorem, which is not essentially different from Mr. Tsuji's, his theorem can be obtained as follows.

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1) $\Omega^{(q)}(M)$ denotes the q-th iteration of $\Omega(M)$.

2) It follows from Landau's expression of $\Omega(M)$ that $M^4 D$ can be used for $\Omega(M)$, when M is larger than a fixed number, D being a numerical constant. By using it for $\Omega(M)$ the right-hand side of (4) becomes

$$M^{4^{13}} D^{1+4+\dots+4^{12}}.$$

c.f. Proc. Phy-Math. Soc. Japan, Ser (3), **8** (1926). 174.

Considering $|f(z_1)-f(0)|=\left|\int_0^{z_1} f'(z)dz\right|=|b-a|$ we must have at least one point η_f on the segment $(\overline{0z_1})$ at which $|f'(\eta_f)|\geqq l_0=\frac{|b-a|}{|z_1|}$ for all the functions (1). In order to apply the above theorem it is necessary to replace (5) by $|C_1|\leqq \Omega^{(q)}(M)/|z_i-\eta_f|$. And from $|\eta_f|\leqq|z_1|$ and $R_i-R_1>|z_1|$, we can obtain an infinite number of concentric ring-regions $R'_i<|z|<R'_{i+1}$, in which all the functions (1) take at least once the value 1 or 0 by taking $R'_{n+1}=R_{2+3n}$.

Similar method admits us to find an expression for an infinite number of radii of concentric ring-regions $R_i<|z|<R_{i+1}$, where all the functions of a class of integral functions $f(z)$, for which $|f^{(m)}(z_0)|\geqq l_0>0$ and $|f(z_i)|<|z_i|^p$, $(i=1,2,\cdots)$, $|z_1|\leqq|z_2|\leqq\cdots\to\infty$, p being a fixed constant, take at least once the value 1 or 0, provided that the integer $m>(4p)^{4^{8\pi}}$.

ERRATA

in my Note: On Some Properties ef Meromorphic Functions. (Vol. 2 (1926) 466–469).

Page 469, line 5 *read* " R_{n+1} " *for* " R_p ".

Page 469, line 7, *read* "[ ]$^{\rho}$" *for* "[ ]".

Page 469, *add* " where $s=x_n$, $p=x_{n-1}$ " *to the end.*