111. On Transcendental Numbers.

By Shin-ichi IZUMI.

Mathematical Institute, Tohoku Imp. University.

(Rec. June 23, 1927. Comm. by M. FUJIWARA, M.I.A., July 12, 1927.)

The following theorem was proved by Kempner.[1]

Let a be an integer greater than 1 ; α_n $(n=0, 1, 2, \ldots\ldots)$ any positive or negative integer smaller in absolute value than a fixed arbitrary number M, but only a finite number of the α_n equal to 0, then

$$f(x)=\sum_{n=0}^{\infty}\frac{\alpha_n}{a_n}x^n, \qquad a_n=a^{2^n},$$

represents a transcendental number for any rational number x.

As Blumberg[2] has shown, the condition that only a finite number of coefficients α_n shall be zero may be removed, so that

$$f_1\left(\frac{p}{q}\right)=\sum\frac{\alpha_{\sigma_n}}{a'_n}\left(\frac{p}{q}\right)^n, \qquad a'_n=a^{2^{\sigma n}}$$

represents a transcendental number, when $\sigma_1<\sigma_2<\ldots\ldots<\sigma_n\to\infty$.

He proved this theorem by distinguishing between two cases, where

(1) for every n there are two consecutive σ_n's greater than n and differing by more than k,

(2) after a certain point, the difference between two consecutive σ_n's is less than or equal to k.

In the following lines I will give a generalzation of Kempner-Blumberg's theorem, which can be proved without distinction of the two cases.

Our theorem runs as follows :

The integral transcendental function

$$f(x)=\sum_{n=0}^{\infty}\frac{\alpha_n}{a^{\sigma_n}}x^n,$$

1) Trans. American Math. Soc., **17** (1916).
2) Bulletin American Math. Soc., **32** (1926).

where a denotes an integer greater than 1 and α_n an integer $< a^n$ in absolute value, represents a transcendental number for any rational x, when the following conditions (A) are satisfied for every k:

$$\lim_{n\to\infty} \frac{\sigma_n}{n} = \infty ,$$

$$\frac{\sigma_{m_1}+\sigma_{m_2}+\ldots\ldots+\sigma_{m_i}}{\sigma_{n_1}+\sigma_{n_2}+\ldots\ldots+\sigma_{n_j}} > 1+\delta_k , \qquad (\delta_k > 0),$$

for $\sigma_{m_1}+\sigma_{m_2}+\ldots\ldots+\sigma_{m_n} > \sigma_{n_1}+{}^{n_2}+\ldots\ldots+\sigma_{n_j}$ where some σ_m's (and also σ_n's) may be equal and $\sigma_m \neq \sigma_n$ $(i, j \leqq k)$, and there is only one set $(\sigma_{n_1}, \sigma_{n_2}, \ldots\ldots, \sigma_{n_i})$ whose sum is largest, but less than σ_n.

To prove this we suppose that $f(p/q)$ is not transcendental, then $z = f(p/q)$ satisfies an algebraic equation with integral coefficients of the form

$$\varphi(z) = \sum_{\mu=0}^{k} A_\mu Z_\mu = 0.$$

We can show that this leads to a contradiction.

The conditions (A) are satisfied for $\sigma_n = 2^n$, so that Kempner-Blumberg's theorem follows immediately.

For $\sigma_n = [r^n]$, $r > 1$, where $[x]$ represents the greatest integer contained in x, the conditions (A) are satisfied for $k=1$. Therefore

$$f_2\left(\frac{p}{q}\right) = \sum \frac{\alpha_n}{a^{[r^n]}} \left(\frac{p}{q}\right)^n$$

represents an irrational number.

When $r > \frac{1+\sqrt{5}}{2}$, the conditions (A) are satisfied for $k=2$. Therefore $f_2(p/q)$ is neither rational, nor a quadratic irrational.

For $r \geqq 2$, $f_2(p/q)$ represents a transcendental number.