PAPERS COMMUNICATED

77. *Differential Geometry of Conics in the Projective Space of Three Dimensions.*

I. *Fundamental Theorem in the Theory of a One-parameter Family of Conics.*

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The elements in the geometry are, in general, arbitrary, so by adopting different elements we can find various new results, which are important and interesting from geometrical points of view. In the differential geometry the theories of lines and of circles (line-geometry and circle-geometry) have been already developed by many authors, but the theory of conics has, up to the present, never been discussed except in some papers dealing with algebraic geometry[1], and remains unsolved after having been proposed by F. Klein in his Erlanger Programm. I will now, in the present note and also in the papers which will appear in the near future, discuss the differential geometry of conics in the projective space of three dimensions R_3. The theory of quadratic cones is completely dual to the present theory in the projective space.

1. *Coordinates of a conics in* R_3. Let the homogeneous coordinates of a plane L in R_3 be l_i $(i=1, 2, 3, 4)$ and consider the homogeneous coordinates of a point on the plane L: x^α $(\alpha=1, 2, 3)$. In that case, we take, as the triangle of reference on the plane L, the triangle formed by three diagonals of the complete quadrilateral, which is the section of the tetrahedron of reference T in R_3 by the plane L. And let the homogeneous point-coordinates in R_3 be y^i, where the tetrahedron of reference is also T. When we represent the coordinates of a point x^α on the plane L by the coordinates in space y^i and inversely the coordinates y^i of a point in space by the coordinates on the plane passing through that point, we get

$$(1)\qquad \begin{cases} \rho y^1=l_2l_3l_4\,(x^1+x^2+x^3), & \rho y^2=l_1l_3l_4\,(-x^1-x^2+x^3), \\ \rho y^3=l_1l_2l_4\,(-x^1+x^2-x^3), & \rho y^4=l_1l_2l_3\,(x^1-x^2-x^3)\,; \end{cases}$$

$$(2)\qquad \begin{cases} \sigma x^1=l_1y^1-l_2y^2-l_3y^3+l_4y^4, & \sigma x^2=l_1y^1-l_2y^2+l_3y^3-l_4y^4, \\ \sigma x^3=l_1y^1+l_2y^2-l_3y^3-l_4y^4, & \end{cases}$$

where

$$(3)\qquad l_iy^i \doteq 0 \quad \text{and} \quad \rho\sigma=4.$$

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1) See R. A. Johnson, The conic as a space element, Trans. of American Math. Soc., **15** (1914); and C.G.F. James, Analytic representation of congruences of conics, Proc. Camb. Phil. Soc. **21** (1922–1923).

Let K be a conic in R_3 and l_i the coordinates of the plane L, on which the conic K lies. If the equation of the conic K

$$(4)\qquad a_{\alpha\beta}x^\alpha x^\beta = 0$$

referred to this coordinate-system on the plane L, then we can consider the coefficients $a_{\alpha\beta}$ in (4) as the homogeneous coordinates of the conic K in the plane L, i.e. conics on L have one-to-one correspondence to the set of numbers $a_{\alpha\beta}$. It may, therefore, be natural to adopt the set of the plane-coordinates l_i and $a_{\alpha\beta}$ together as the (doubly) homogeneous coordinates of a conic in R_3[1].

2. *Projective transformations.* By a projective transformation the plane-coordinates are in general transformed, as we know, by the following rule:

$$(5)\qquad l_i^* = P_i^j l_j,$$

i.e. a linear transformation. On the other hand the homogeneous coordinates $a_{\alpha\beta}$ of a conic in the plane L are transformed by a projective transformation on L as follows:

$$(6)\qquad a^*_{\alpha\beta} = Q^\gamma_\alpha Q^\delta_\beta a_{\gamma\delta}.$$

In general, a projective transformation in R_3 is decomposed into two transformations, one of which makes the plane L invariant and the other transforms the plane L into another plane. We can, therefore, see that the projective transformation of the above-mentioned coordinates of conics is represented by both (5) and (6) considered at the same time.

3. *The projective length.* Let a one-parameter family of conics in R_3 be represented in parametric form by

$$(7)\qquad a_{\alpha\beta} = a_{\alpha\beta}(t), \qquad l_i = l_i(t)$$

or by vector-notation

$$(7)'\qquad \mathfrak{a} = \mathfrak{a}(t), \qquad \mathfrak{l} = \mathfrak{l}(t),$$

where we assume for brevity that all the functions are analytic, and that $a_{\alpha\beta}$ are normalized by the relation

$$(8)\qquad 6\,|a_{\alpha\beta}| \equiv (\mathfrak{a}, \mathfrak{a}, \mathfrak{a}) = 1,$$ [2]

then we get by differentiation

$$(9)\qquad \frac{d}{dt}(\mathfrak{a}, \mathfrak{a}, \mathfrak{a}) = 3\left(\frac{d\mathfrak{a}}{dt}, \mathfrak{a}, \mathfrak{a}\right) = 0.$$

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1) On various other coordinates of a conic in R_3 see: Spottiswoode, Proc. of London Math. Soc., 1878, 185–196; Reye, Crelles Journal, **82** (1876), 54–83; P. van Geer, Archieves Néerlandaises, 1888, 58–90; Godeaux, Bulletin de l'Acad. de Belgique, 1908, 896–902.

2) Concerning the notation $(\mathfrak{a}, \mathfrak{a}, \mathfrak{a})$ see my previous paper, Fundamental forms in the projective differential geometry of m-parametric families of hypersurfaces of the second degree in the n-dimensional space, these Proceedings, **3** (1927), 311.

But we can now see that the quantity

$$(10)\qquad \left(\frac{d\mathfrak{a}}{dt}, \frac{d\mathfrak{a}}{dt}, \mathfrak{a}\right) = -\frac{1}{2}\left(\frac{d^2\mathfrak{a}}{dt^2}, \mathfrak{a}, \mathfrak{a}\right)$$

is, in general, different from zero. Hence we adopt the new variable

$$(11)\qquad \sigma = i\int\left(\frac{d\mathfrak{a}}{dt}, \frac{d\mathfrak{a}}{dt}, \mathfrak{a}\right)^{\frac{1}{2}} dt, \qquad i=\sqrt{-1}$$

as the natural parameter and call it the projective length of the family of conics.

N.B. As natural parameter we may take the so-called projective length[1] of the developable surface generated by the plane $\mathfrak{l}$.

4. *The differential invariants.* As I have shown in one of my previous papers[2], we know that the fundamental differential invariants with regard to $\mathfrak{a}$ are only

$$(12)\qquad I_1=(\mathfrak{a}', \mathfrak{a}', \mathfrak{a}'),\quad I_2=(\mathfrak{a}'', \mathfrak{a}'', \mathfrak{a}),\quad I_3=(\mathfrak{a}'', \mathfrak{a}'', \mathfrak{a}'),\quad I_4=(\mathfrak{a}'' \mathfrak{a}'', \mathfrak{a}''),$$

$$I_5 = \sum_{\alpha,\beta,\gamma,\delta,\varepsilon,\tau} A^{(\mathfrak{a})}_{\alpha\beta} A^{(\mathfrak{a}')}_{\gamma\delta} A^{(\mathfrak{a}'')}_{\varepsilon\tau}\, a_{\delta\varepsilon}\, a'_{\tau\alpha}\, a''_{\beta\gamma}$$ [3]

and between these five invariants there exists a relation of the third degree with regard to I_5. Hence one of these quantities is not essential.

Next normalize $\mathfrak{l}$ so that the determinant

$$(13)\qquad |l\, l'\, l''\, l'''| = 1,$$

then we get

$$(14)\qquad |l\, l'\, l''\, l^{IV}| = 0.$$

Put

$$(15)\qquad |l\, l'\, l'''\, l^{IV}| = 6p,\qquad |l\, l''\, l'''\, l^{IV}| = -4q,\qquad |l'\, l''\, l'''\, l^{IV}| = r.$$

and

$$(16)\qquad \begin{cases} J_1 = -\dfrac{3}{5}p, & J_2 = q - \dfrac{3}{2}\dfrac{dp}{d\sigma}, \\ J_3 = r - 2\dfrac{dq}{d\sigma} + \dfrac{6}{5}\dfrac{d^2p}{d\sigma^2} - \dfrac{81}{25}p^2, & \end{cases}$$

or

$$(17)\qquad p = -\frac{5}{3}J_1,\qquad q = J_2 + \frac{5}{2}J_1',\qquad r = J_3 + 3J_1'' + 9J_1^2 + 2J_2'.$$

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1) See G. Sannia, Nuova trattazione della geometria proiettivo-differenziale delle curve sghembe I, II, Annali di Matematica, ser. 4, **1**, **3** (1923–25).

2) A. Kawaguchi, Ueber projektive Differentialgeometrie I. Theorie der Kegelschnittscharen in der Ebene, Tohoku Math. Journ., **28** (1927), 126–146.

3) Concerning the notation $A_{\alpha\beta}$ see my paper, Fundamental forms etc., loc. cit.

The differential invariants J_ν ($\nu=1, 2, 3$) are the fundamental ones with respect to $\mathfrak{l}$, i.e. the developable surface adjoining to our family of conics. The order of these fundamental differential invariants is

$$(I_1, 1),\ (I_2, 2)\ (I_3, 2),\ (I_4, 2),\ (I_5, 2),\ (J_1, 5),\ (J_2, 6),\ (J_3, 7).$$

5. *The fundamental theorem.* We can now prove the fundamental theorem:

When the invariants I_λ ($\lambda=1, 2, 3, 4, 5$) and J_ν ($\nu=1, 2, 3$) *are given as functions of the projective length* σ, *where the relation between* I_λ *(above mentioned) holds good, then the one-parameter family of conics in* R_3 *is uniquely de'ermined, except for projective transformations.*

6. *Other connected problems.* The generating lines G_1 of the developable surface $\mathfrak{l}$ are given by $l_i y^i = 0,\ l_i' y^i = 0$, i.e.

$$\lambda_\alpha x^\alpha = 0, \tag{18}$$

where

$$\begin{cases} \lambda_1 = l_1' l_2 l_3 l_4 - l_1 l_2' l_3 l_4 - l_1 l_2 l_3' l_4 + l_1 l_2 l_3 l_4', \\ \lambda_2 = l_1' l_2 l_3 l_4 - l_1 l_2' l_3 l_4 + l_1 l_2 l_3' l_4 - l_1 l_2 l_3 l_4', \\ \lambda_3 = l_1' l_2 l_3 l_4 + l_1 l_2' l_3 l_4 - l_1 l_2 l_3' l_4 - l_1 l_2 l_3 l_4'. \end{cases} \tag{19}$$

Moreover the points P_1 on the edge of regression are represented by

$$\begin{cases} x^{*1} = \rho \begin{vmatrix} l_1' l_4 - l_1 l_4' & l_2' l_3 - l_2 l_3' \\ l_1'' l_4 - l_1 l_4'' & l_2'' l_3 - l_2 l_3'' \end{vmatrix}, \\ x^{*2} = \rho \begin{vmatrix} l_1' l_3 - l_1 l_3' & l_4' l_2 - l_4 l_2' \\ l_1'' l_3 - l_1 l_3'' & l_4'' l_2 - l_4 l_2'' \end{vmatrix}, \qquad -\rho = \prod_{i=1}^{4} l_i, \\ x^{*3} = \rho \begin{vmatrix} l_1' l_2 - l_1 l_2' & l_3' l_4 - l_3 l_4' \\ l_1'' l_2 - l_1 l_2'' & l_3'' l_4 - l_3 l_4'' \end{vmatrix}, \end{cases} \tag{20}$$

the poles P_2 of G_1 with regard to the conic $\mathfrak{a}$ by

$$x^1 = \begin{vmatrix} \lambda_1 & a_{12} & a_{13} \\ \lambda_2 & a_{22} & a_{23} \\ \lambda_3 & a_{32} & a_{33} \end{vmatrix}, \quad x^2 = \begin{vmatrix} a_{11} & \lambda_1 & a_{13} \\ a_{21} & \lambda_2 & a_{23} \\ a_{31} & \lambda_3 & a_{33} \end{vmatrix}, \quad x^3 = \begin{vmatrix} a_{11} & a_{12} & \lambda_1 \\ a_{21} & a_{22} & \lambda_2 \\ a_{31} & a_{32} & \lambda_3 \end{vmatrix}, \tag{21}$$

and the polar lines G_2 of P_1 by $a_{\alpha\beta} x^{*\alpha} x^\beta = 0$. The properties of the curves and ruled surfaces traced by these points and lines are desirable to be discussed in our theory, also the detailed theory of special families of conics, e.g. the minimal family: $(\mathfrak{a}', \mathfrak{a}', \mathfrak{a}) \equiv 0$, the family with constant invariants, etc. remains yet untouched. I hope to make these investigations in another papers.
