PAPERS COMMUNICATED

156. *On the Singularity of the Functions Defined by Dirichlet's Series.*

By Shin-ichi IZUMI.
Mathematical Institute, Tohoku Imperial University, Sendai.

(Comm. by M. FUJIWARA, M.I.A., Nov. 2, 1928.)

The object of this paper is to extend Vivanti's theorem and its generalizations to functions defined by Dirichlet's series.

1. Let $r_1, r_2, r_3, \ldots$ be a sequence of real numbers such that

$$0<r_1<r_2<r_3<\ldots\ldots, \quad \frac{r_\nu}{\nu}\to\infty.$$

Then the integral function

$$G(z)=\prod_{\nu=1}^{\infty}\left(1-\frac{z^2}{r_\nu^2}\right)^2 \tag{1.1}$$

is of order 1 and of minimal type. Let us next consider the Dirichlet's series:

$$D(s)=\sum_{\nu=1}^{\infty}c_{\lambda_\nu}e^{-\lambda_\nu s}\quad\left(\begin{matrix}c_{\lambda_\nu}=a_{\lambda_\nu}+ib_{\lambda_\nu}\\ 0\leqq\lambda_1<\lambda_2<\lambda_3<\ldots\ldots, \quad \lambda_\nu\to\infty\end{matrix}\right). \tag{1.2}$$

Then we have

Lemma 1. *The Dirichlet's series*

$$H(s)=\sum_{\nu=1}^{\infty}c_{\lambda_\nu}G(\lambda_\nu)e^{-\lambda_\nu s} \tag{1.3}$$

and (1.2) *have the same convergence abscissa, when*

$$\varliminf_{\nu=\infty}(\lambda_\nu-\lambda_{\nu-1}), \quad \varliminf_{\varkappa,\nu=\infty}(r_\varkappa-\lambda_\nu)>0. \tag{1.4}$$

After Dr. Cramér[1] (1.3) converges at least in the domain, where (1.2) is convergent. So it suffices to prove the converse. To this purpose we will first calculate the order of $G(\lambda_\mu)$.

Let n be an integer such that $r_n<\lambda_\mu<r_{n+1}$. By (1.4) we have then $r_\nu-r_{\nu-1}>h$, $r_\varkappa-\lambda_\nu>h$ for all ν and $\varkappa$. In general we can suppose that $h=1$.

Now[2]
$$\frac{1}{G(\lambda_\mu)}\leqq\prod_{\nu=1}^{n}\frac{1}{\left(\frac{\lambda_\mu}{r_\nu}-1\right)^2}\prod_{\nu=n+1}^{\infty}\left(1+\frac{\lambda_\mu^2}{(r_\nu+\lambda_\mu)(r_\nu-\lambda_\mu)}\right)^2$$
$$\leqq\frac{\lambda_\mu^{2n}}{(n!)^2}\prod_{\nu=n+1}^{\infty}\left(1+\frac{\lambda_\mu^2\varepsilon_\nu}{\nu(\nu-n)}\right)^2\quad\left(\varepsilon_\nu=\frac{\nu}{r_\nu}<\varepsilon^2\right)$$

---

1) Cramér, Arkiv för Math. **13** (1919).
2) See Carlson u. Landau, Göttinger Nachrichten, 1921.

$$< \frac{\lambda_\mu^{2n}}{n^{2n}} \cdot \prod_{\nu=n+1}^{\infty} \left(1+\left(\frac{\varepsilon\lambda_\mu}{\nu}\right)^2\right)^2$$

$$= \left(\frac{\sin \pi i \lambda_\mu \varepsilon}{\pi i \lambda_\mu \varepsilon}\right)^2 \cdot e^{2e\lambda_\mu \cdot \frac{n}{e\lambda_\mu} \log \frac{e\lambda_\mu}{n}} < C e^{\delta\lambda_\mu}.$$

Suppose that (1.3) is convergent for $\sigma > l$, then

$$A_\nu = \sum_{\mu=1}^{\nu} c_{\lambda_\mu} G(\lambda_\mu) = O\left(e^{\lambda_\nu (l+\varepsilon)}\right).$$

And $$\left|\sum_{\nu=1}^{n} c_{\lambda_\nu}\right| < \left|\sum_{\nu=1}^{n} c_{\lambda_\nu} G(\lambda_\nu) \frac{1}{G(\lambda_\nu)}\right| = \left|\sum_{\nu=1}^{n-1} A_\nu \left(\frac{1}{G(\lambda_\nu)} - \frac{1}{G(\lambda_{\nu+1})}\right) + \frac{A_n}{G(\lambda_n)}\right|$$

$$< \operatorname*{Max}_{1 \leq \nu < n} |A_\nu| \cdot \sum_{\nu=1}^{n-1} \left|\frac{1}{G(\lambda_\nu)} - \frac{1}{G(\lambda_{\nu+1})}\right| + \frac{|A_n|}{G(\lambda_n)},$$

where $$\sum_{\nu=1}^{n-1} \left|\frac{1}{G(\lambda_\nu)} - \frac{1}{G(\lambda_{\nu+1})}\right| = \sum_{\nu=1}^{n-1} \frac{|G(\lambda_{\nu+1}) - G(\lambda_\nu)|}{G(\lambda_\nu) G(\lambda_{\nu+1})}$$

$$< e^{2\varepsilon\lambda_n} \cdot \sum_{\nu=1}^{n-1} |G(\lambda_{\nu+1}) - G(\lambda_\nu)| < e^{2\varepsilon\lambda_n} \int_0^{\lambda_n} |G'(x)|\, dx < e^{4\varepsilon\lambda_n},$$ [1]

so that $$\sum_{\nu=1}^{n} c_{\lambda_\nu} = O\left(e^{\lambda_n (l+\varepsilon')}\right).$$

That is, (1.2) is convergent for $\sigma > l$. q.e.d.

2. Consider the Dirichlet's series with real coefficients:

$$f(s) = \sum_{\nu=1}^{\infty} a_{\lambda_\nu} e^{-\lambda_\nu s}, \tag{2.1}$$

whose convergence abscissa is finite, for example $\sigma = 0$. From[2] the sequence (λ_ν) select a subsequence (r_ν) such that

$$\frac{r_\nu}{\nu} \to \infty \quad \text{and} \quad \varliminf_{\nu=\infty} (r_\nu - r_{\nu-1}), \quad \varliminf_{\varkappa,\nu=\infty} (r_\nu - \lambda_\varkappa) > 0.$$

Let (μ_ν) be the complementary sequence of (r_ν), then we have

$$f(s) = \sum_{\nu=1}^{\infty} a_{r_\nu} e^{-r_\nu s} + \sum_{\nu=1}^{\infty} a_{\mu_\nu} e^{-\mu_\nu s} = g(s) + h(s) \quad \text{say}.$$

We will now distinguish two cases. First let the convergence abscissa of $h(s)$ be greater than 0, then that of $g(s)$ is 0. In this case the point $s=0$ is a singular point of $g(s)$, as the Carlson-Landau-Szász's theorem[3] shows us, so that $s=0$ is also a singular point of $f(s)$. Next

1) Cf. Cramér, loc. cit.
2) Carlson u. Landau, loc. cit.; Szász, Math. Ann. **85** (1922).
3) Landau, Math. Ann. **61** (1905).

let the convergence abscissa of $h(s)$ be $\sigma=0$. By Lemma 1 the convergence abscissa of

$$\sum_{\nu=1}^{\infty} a_{\mu_\nu} G(\mu_\nu) e^{-\mu_\nu s} = \sum_{\nu=1}^{\infty} a_{\lambda_\nu} G(\lambda_\nu) e^{-\lambda_\nu s} \tag{2.2}$$

is $\sigma=0$. If we suppose that $a_{\mu_\nu} \geqq 0$ for all ν, that is

$$a_{\lambda_\nu} \geqq 0 \tag{2.3}$$

with the exception of a_{r_ν}, which is arbitrary, then we have

$$a_{\lambda_\nu} G(\lambda_\nu) \geqq 0 \tag{2.4}$$

for all ν. So by the Landau's theorem[1] $s=0$ is a singular point of (2.2). On the other hand Dr. Cramér[2] proved that (2.2) has no singularities other than those of (2.1). It follows that $s=0$ is a singular point of (2.1). Thus we have established the following

Lemma 2. *Let the Dirichlet's series with real coefficients* (2.1) *have the finite convergence abscissa $\sigma=a$, and $a_{\lambda_\nu} \geqq 0$ except (a_{r_ν}) which are real or complex, and*

$$\frac{r_\nu}{\nu} \to \infty, \quad \varliminf_{\nu=\infty} (r_\nu - r_{\nu-1}), \quad \varliminf_{\varkappa,\nu=\infty} (r_\nu - \mu_\varkappa) > 0\,.$$

Then $f(s)$ is singular at $s=a$.

This is a generalization of the Landau's theorem.[3]

3. Let us now proceed to our principal theorem. Take a general Dirichlet's series (1.2), whose convergence abscissa is finite $\sigma=a$, and consider

$$\sum_{\nu=1}^{\infty} a_{\lambda_\nu} e^{-\lambda_\nu s} \quad \text{and} \quad \sum_{\nu=1}^{\infty} b_{\lambda_\nu} e^{-\lambda_\nu s} \tag{3.1}$$

Then at least one of (3.1) has the same convergence abscissa as (1.2). Let us suppose that $a_{\mu_\nu}, b_{\mu_\nu} \geqq 0$. Then $\sigma=a$ is a singular point of at least one of (3.1), so that this point is also singular for (1.2)[4]. Thus we get the following

Theorem 1. *Suppose that the Dirichlet's series* (1.2) *has the finite convergence abscissa, $\sigma=a$, and $0 \leqq \arg c_{\lambda_\nu} \leqq \frac{\pi}{2}$ with the exception of c_{r_ν} such that*

$$\frac{r_\nu}{\nu} \to \infty, \quad \varliminf_{\nu=\infty} (r_\nu - r_{\nu-1}), \quad \varliminf_{\varkappa,\nu=\infty} (r_\nu - \lambda_\varkappa) > 0\,.$$

Then $s=a$ is a singular point of the function defined by (1.2).

---

1) Cramér, loc. cit.
2) Landau, loc. cit.
3) Szász, loc. cit.
4) Kojima, Tohoku Math. Journ. **17** (1918).

4. Suppose that the conditions in the theorem are satisfied and that $\lim_{n=\infty} e^{2\pi i\mu_n\varphi}=e^{2\pi i\psi}$ for some irrational number φ. Let us consider the series

$$(4.1)\qquad \sum_{n=0}^{\infty} c_{\lambda_n} G(\lambda_n)\, e^{2\pi i\lambda_n\varphi p}\cdot e^{-\lambda_n s}=\sum_{n=0}^{\infty} c_{\mu_n} G(\mu_n) e^{2\pi i\mu_n\varphi p}\, e^{-\mu_n s},$$

where p is a positive integer. Multiplying a constant we get

$$(4.2)\qquad -i\sum_{n=0}^{\infty} c_{\mu_n} G(\mu_n)\, e^{2\pi i\psi_n}\, e^{-\mu_n s}\quad \left(\psi_n=\mu_n p\varphi-p\psi+\frac{1}{8}\right).$$

At least one of the series

$$(4.3)\qquad \sum_{n=1}^{\infty} a_{\mu_n} G(\mu_n) e^{2\pi i\mu_n p\varphi}\, e^{-\mu_n s}\quad\text{and}\quad \sum_{n=0}^{\infty} b_{\mu_n} G(\mu_n)\, e^{2\pi i\mu_n p\varphi}\, e^{-\mu_n s}$$

must have the same convergence abscissa as (4.1). For definiteness suppose the first to be true. Then, as easily to be seen from the Kojima's theorem,[1)]

$$\sum_{n=0}^{\infty} R\left(-ic_{\mu_n} G(\mu_n) e^{2\pi i\psi_n}\right) e^{-\mu_n s}$$

has the same convergence abscissa as (4.3). By Theorem 1 $s=a$ is a singular point of (4.2). That is, the points

$$(4.4)\qquad s=a+(p'\varphi+2n\pi)i\quad (p\equiv p' (\mathrm{mod}\ 2\pi)\,;\ \ p,\ n=1,\ 2,\ \ldots\ldots)$$

are singular points of (4.1) and then of (1.2). Since the point set (4.4) is everywhere dense on the convergence line $\sigma=a$, this line is the singular line. So we have

Theorem 2. *Suppose that the Dirichlet's series* (1.2) *has the finite convergence abscissa* $\sigma=a$, *and* $0\leqq \arg c_{\lambda_\nu}\leqq\frac{\pi}{2}$ *with the exception of* c_{r_ν} *such that*

$$\frac{r_\nu}{\nu}\to\infty,\quad \varliminf_{\nu=\infty}(r_\nu-r_{\nu-1}),\quad \varliminf_{\nu,\kappa=\infty}(r_\nu-\lambda_\kappa)>0,$$

suppose further that $\lim_{\nu=\infty} e^{2\pi i\mu_\nu\varphi}$ *exists for some irrational number* φ *and for the complementary set* (μ_ν) *of* (r_ν). *Then the series* (1.2) *has the convergence line as the singular line.*

This is a generalization of Gergen-Widder's theorem.

---

1) Gergen-Widder, Am. Journ. of Math. **50** (1928).