38. *On Hankel Transforms.*

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1. The notion of the Fourier transforms arises from Fourier's integral formula

$$(1) \qquad f(x)=\frac{2}{\pi}\int_0^\infty \cos x\,u\,du\int_0^\infty \cos x\,t\,f(t)\,dt,$$

which gives the reciprocal relation

$$(2) \quad f(x)=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\infty \cos x\,u\,F(u)\,du\,, \quad F(x)=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\infty \cos x\,u f(u)\,du$$

between two functions $f(x)$ and $F(x)$. Each one of two functions so related is said to be the Fourier transform of the other. When we study the relations (2), it is desirable to find any theorem of the form: "when $f(x)$ satisfies certain conditions, so does $F(x)$, and the reciprocity holds goods." For this purpose the reciprocal relations above stated were tranformed into the from

$$(3) \quad f(x)=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{d}{dx}\int_0^\infty \frac{\sin xu}{u}F(u)du\,, \quad F(x)=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{d}{dx}\int_0^\infty \frac{\sin xu}{u}f(u)du\,,$$

which reduce to (2), when differentiation under the sign of integration is permissible. Then the following theorem was established.

If $\int_0^\infty (f(x))^2\,dx$ exists, then $\int_0^\infty (F(x))^2\,dx$ exists, and the reciprocity holds. Further we have $\int_0^\infty (f(x))^2\,dx=\int_0^\infty (F(x))^2\,dx$.

This was gotten by Prof. Plancherel.[1] He deduced this theorem as a special case of the reciprocal relation belonging to a general type of functional transformation. But the deduction is by no means immediate. Lately Mr. Titchmarsh[2] proved this theorem in direct way, and Prof. Hardy[3] gave an alternate proof.

1) Plancherel: Rendiconti di Palermo, **30** (1910).
2) Titchmarsh: Proc. Cambridge Phil. Soc., **21** (1924).
3) Hardy: Messenger of Mathematics, **48** (1925).

2. The reciprocal relations of a more general character than Fourier's can be derived from Hankel's integral formula,

$$f(x)=\int_0^\infty J_\nu(ux)\,u\,du\int_0^\infty J_\nu(ut)\,t f(t)\,dt\,.$$

We have, in fact, writing $x^{-\frac{1}{2}}f(x)$ for $f(x)$,

$$(4)\quad f(x)=\int_0^\infty (ux)^{\frac{1}{2}}J_\nu(ux)\,F(u)\,du\,,\quad F(x)=\int_0^\infty (ux)^{\frac{1}{2}}J_\nu(ux)\,f(u)\,du\,.$$

Each one of two functions connected by relations of this nature is called the Hankel transform of the other. In the case $\nu=-\frac{1}{2}$, Hankel transforms reduce to Fourier transforms. Mr. Titchmarsh[1] transformed this reciprocal transformation, as we did for (2), into the form

$$(5)\quad \begin{aligned} f(x)&=x^{-\nu-\frac{1}{2}}\frac{d}{dx}\left\{x^{\nu+1}\int_0^\infty u^{-\frac{1}{2}}J_{\nu+1}(ux)\,F(u)\,du\right\},\\ F(x)&=x^{-\nu-\frac{1}{2}}\frac{d}{dx}\left\{x^{\nu+1}\int_0^\infty u^{-\frac{1}{2}}J_{\nu+1}(ux)\,f(u)\,du\right\},\end{aligned}$$

which reduce to (4) when the differentiation under the sign of integration is permissible. And he proved the theorem in §1, for the functions $f(x)$ and $F(x)$ connected by (5).

3. Recently Prof. F. Riesz[2] proved very elegantly the theorem in §1, for the functions $f(x)$ and $F(x)$ connected by (2) instead of (3). But, in this case, it was supposed that $f(x)$ and $F(x)$ are integrable in $(0,\infty)$. By Prof. Fujiwara's suggestion, the author proved a theorem for the functions $f(x)$ and $F(x)$ connected by (4), instead of (5), by the method due to Prof. F. Riesz.

4. To prove this we consider the integral

$$(6)\quad I=\iiint_0^\infty e^{-\frac{u^2}{2n^2}}\sqrt{ut}\,J_\nu(ut)\cdot\sqrt{ux}\,J_\nu(ux)f(t)\,g(x)\,du\,dt\,dx.$$

If we suppose that $f(t)$ and $g(x)$ are integrable in $(0,\infty)$, we can invert the order of integration as we like.

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1) Titchmarsh: loc. cit.

2) Riesz: Acta de Szeged, **3** (1928).

Now we put $g(x)=f(x)$. Then by the integration of I in the order u, t, x we get

$$(7)\quad |I|=\left| n^2 e^{-\frac{\nu\pi i}{2}} \int_0^\infty\int_0^\infty \sqrt{xt}\, e^{-\frac{x^2+t^2}{2}n^2} J_\nu(n^2xti)\,(f(t))^2\,dt\,dx \right|$$

$$\leqq n^2\int_0^\infty (f(t))^2\,dt \int_0^\infty \sqrt{xt}\, e^{-\frac{x^2+t^2}{2}n^2} |J_\nu(n^2xti)|\,dx = \int_0^\infty (f(t))^2 dt + o(1)$$

taking the Schwarz's inequality into account. The integration in the order u, t, x gives

$$(8)\qquad I=\int_0^\infty e^{-\frac{u^2}{2n^2}}(F(u))^2\,du\,.$$

Thus we have, by (7) and (8) and applying the Fatou's theorem,

$$\int_0^\infty (F(u))^2\,du \leqq \int_0^\infty (f(t))^2\,dt\,.$$

Putting $g(x)=1$ in $(0, a)$; $=0$ in (a, ∞) in (6), we can prove that the transform of $F(u)$ is $f(t)$. Then we get

$$\int_0^\infty (f(t))^2\,dt \leqq \int_0^\infty (F(u))^2\,du\,.$$

Hence we have

$$\int_0^\infty (f(t))^2\,dt = \int_0^\infty (F(u))^2\,du\,.$$