35. *On the Theory of Meromorphic Functions.*

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Let $w=f(z)$ be a meromorphic function in the whole finite z-plane. Now consider the function depending on $|z|=r$ and $f(z)$:

$$A(r,f)=\frac{1}{\pi}\int_0^r\int_0^{2\pi}\frac{|(\rho f'e^{i\theta})|^2}{(1+|f(\rho e^{i\theta})|^2)^2}\rho d\rho d\theta\,, \quad \ldots\ldots\ldots\ldots\ldots (1)$$

which is the area of the domain mapped by $w=f(z)$ for $|z|\leqq r$, and projected on the Riemann sphere of radius $\frac{1}{2}$ touching the w-plane at the origin, divided by the whole area of the Riemann sphere; that is, a mean number of sheets of the Riemann surface of the inverse function of $f(z)$ in $|z|<r$.

By the identity, which holds in the domain where $f(z)$ is regular:

$$\frac{4|f'(z)|)^2}{(1+|f(z)|^2)^2}=\Delta\log(1+|f(z)|^2)\,,$$

and Green's transformation formula in the domain $|z|\leqq r$, excluding small circles about the poles of $f(z)$ in $|z|<r$:

$$\iint(u\Delta v-v\Delta u)\,d\sigma=\int\left(u\frac{\partial v}{\partial n}-v\frac{\partial u}{\partial n}\right)ds\,,$$

we obtain, putting $u\equiv 1,\ v\equiv\log(1+|f(z)|^2)$,

$$A(r,f)=\frac{1}{4\pi}\int_0^{2\pi}\frac{\partial\log(1+|f(re^{i\theta})|^2)}{\partial r}rd\theta+n(r,\infty)\,,$$

when $n(r,\infty)$ denotes the number of the poles of $f(z)$ in $|z|<r$.

Putting $\quad b(r,f)=\dfrac{1}{4\pi}\displaystyle\int_0^{2\pi}\frac{\partial\log(1+|f(re^{i\theta})|^2)}{\partial r}rd\theta\,,$

we have obtained the following theorems.

Theorem I. $\quad A(r,f)=b(r,f)+n(r,\infty)\,. \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$

Theorem II. $A(r,f)$ *is a continuous positive increasing function of* r.

Dividing by r and integrating with respect to r the right hand side of (2) from $\varepsilon>0$ to r we have

Theorem III. $$T(r,f)=\int_{\varepsilon}^{r}\frac{A(\rho,f)}{\rho}d\rho+O(1)\,, \quad\ldots\ldots\ldots\ldots\ldots (3)$$

and $$m(r,f)=\int_{\varepsilon}^{r}\frac{b(\rho,f)}{\rho}d\rho+O(1)\,,$$

where $T(r,f)$ and $m(r,f)$ denote the functions introduced by R. Nevanlinna in his researches on the theory of meromorphic functions.

We have in (3) a remarkable relation between the growth of $f(z)$ and a mean number of sheets of the Riemann surface of the inverse function of $f(z)$ in $|z|\leqq r$.

The functions $T_1(r,f)\equiv\int_{\varepsilon}^{r}\frac{A(\rho,f)}{\rho}d\rho$ and $m_1(r,f)\equiv\int_{\varepsilon}^{r}\frac{b(\rho,f)}{\rho}d\rho$ play similar rolls as $T(r,f)$ and $m(r,f)$ respectively.

The following theorems can be obtained from (3):

Theorem IV. *If $f(z)$ is a rational function, then $A(r,f)<M$, where M is a constant, and conversely.*

Corollary. If $f(z)$ is a ratinal function, then $A(r,f)$ tends to a positive integer, when $r\rightarrow\infty$.

Theorem V. *For a meromorphic function of finite order*

$$\overline{\lim_{r=\infty}}\frac{\log T(r,f)}{\log r}=\overline{\lim_{r=\infty}}\frac{\log A(r,f)}{\log r}\,.$$

By our function $A(r,f)$ we can define the order of a meromorphic function.

By Nevanlinna's theorems[1] and a modified theorem of Borel on the growth of a continuous function we have obtained

Theorem VI. *For a decreasing $\varepsilon(r)\rightarrow 0$ for $r\rightarrow\infty$*

$$A(r,f)^{1+\varepsilon(r)}>n(r,a)\,,$$

except possibly in a sequence of intervals where the total variation of loglog r *is finite.*

Theorem VII. *For three values a_1, a_2 and a_3, different from each other,*

$$A(r,f)^{1-\varepsilon(r)}<\sum_{\nu=1}^{3}n(r,a_\nu)\,,$$

except possibly in a sequence of intervals as in Theorem VI.

1) Acta mathematica **46** (1925), 1.

Theorem VIII.[1] *For a sequence of infinite number of intervals tending to* ∞ *we have for* q *values* a_ν ($\nu=1, 2, \dots q$), *different from each other,*

$$(1-\varepsilon)(q-2)A(r,f) < \sum_{\nu=1}^{q} n(r, a_\nu)\,. \qquad \dots\dots\dots\dots \quad (4)$$

Further:

Theorem IX. *For any small* ε *and for four constants* α, β, γ *and* δ

$$(1+\varepsilon)A(r,f) \geqq A\left(r, \frac{\alpha f+\beta}{\gamma f+\delta}\right) \geqq (1-\varepsilon)A(r,f)\,,$$

except in a sequence of intervals where the total variation of $\log r$ *is finite.*

Considering $f(z)$ as a quotient $\frac{\pi_1(z)}{\pi_2(z)}$ of integral functions, where $\pi_2(z)$ is a canonical product of primary factors with respect to the poles of $f(z)$, of possibly least genus, defined by Borel, or by Denjoy for a meromorphic function of infinite order, we define by

$$\mathfrak{M}(r,f) \equiv \operatorname{Max}\sqrt{|\pi_1(re^{i\theta})|^2+|\pi_2(re^{i\theta})|^2}$$

the maximum order of a meromorphic function $f(z)$ for $|z|=r$, which is an extension of the maximum modulus of an integral function.

We have obtained

Theorem X. *For* $R > r$ *we have*

$$T(r,f) \leqq \log\mathfrak{M}(r,f)+O(1)$$
$$\leqq \frac{R+r}{R-r}\left\{T(R,f)+\frac{4Rr}{(R+r)^2}m\left(R, \frac{1}{\pi_2}\right)+O(1)\right\}.$$

Whence

Theorem XI. *For a meromorphic function of finite order*

$$\overline{\lim_{r=\infty}} \frac{\log\log\mathfrak{M}(r,f)}{\log r} = \overline{\lim_{r=\infty}} \frac{\log T(r,f)}{\log r}\,.$$

We can also define by $\log\mathfrak{M}(r,f)$ the order of a meromorphic function. Of course $\mathfrak{M}(r,f)$ is a monotonously increasing function or r, and $\log\mathfrak{M}(r,f)$ a convex function of $\log r$.

The above theorems can be so extended as to apply to a function having an isolated essential singular point, in the neighbourhood of which the function is uniform and meromorphic.

We can also consider $A(r,f)$ and $b(r,f)$ for a function which is meromorphic in the unit-circle[2].

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1) A. Bloch has remarked that (4) will probably hold good. c.f. L'enseignement mathématique, **25** (1926), 94.

2) The detailed proofs of the above theorems and allied theorems will appear in the Japanese Journal of Mathematics, **6** (1929).