90. *Analytic Proof of Blaschke's Theorem on the Curve of Constant Breadth, II.*

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In the former paper with the same title, this Proceedings **3**, 1927, I have given an analytic proof of Blaschke's theorem:

The Reuleaux triangle consisting of three circular arcs of radius a is a curve of constant breadth a with minimum area.

There I have only sketched the main line of proof and left untouched the proof of the fact, that we can determine A and B such that

$$L(\theta)+a\leqq 0 \qquad \text{for} \qquad 0\leqq\theta<\frac{\pi}{3},$$

$$L(\theta)+a\cos\left(\frac{\pi}{3}-\theta\right)\geqq 0 \qquad \text{for} \qquad \frac{\pi}{3}\leqq\theta<\frac{2\pi}{3},$$

$$L(\theta)+a(1+\cos\theta)\leqq 0 \qquad \text{for} \qquad \frac{2\pi}{3}\leqq\theta<\pi,$$

where $$L(\theta)=\int_0^\theta\rho(\varphi)\sin(\theta-\varphi)d\varphi+A\cos\theta+B\sin\theta-a.$$

When I recently informed my proof to Mr. Morimoto, he remarked me a slight error in it. So I will give here the corrected proof in detail.

Determining A and B such that

$$0=L(\theta)+a=L(\theta)+a\cos\left(\frac{\pi}{3}-\theta\right) \qquad \text{for} \qquad \theta=\frac{\pi}{3},$$

$$0=L(\theta)+a\cos\left(\frac{\pi}{3}-\theta\right)=L(\theta)+a(1+\cos\theta) \qquad \text{for} \qquad \theta=\frac{2\pi}{3},$$

and putting these values in $L(\theta)$, we get

$$L(\theta)=-a-\frac{a}{\sqrt{3}}\sin\left(\frac{\pi}{3}-\theta\right)+\int_0^\theta\rho(\varphi)\sin(\theta-\varphi)d\varphi$$
$$+\frac{2}{\sqrt{3}}\sin\left(\frac{\pi}{3}-\theta\right)\int_0^{\frac{2\pi}{3}}\rho(\varphi)\sin\left(\frac{2\pi}{3}-\varphi\right)d\varphi$$
$$-\frac{2}{\sqrt{3}}\sin\left(\frac{2\pi}{3}-\theta\right)\int_0^{\frac{\pi}{3}}\rho(\varphi)\sin\left(\frac{\pi}{3}-\varphi\right)d\varphi.$$

In the case $\frac{\pi}{3}<\theta<\frac{2\pi}{3}$, we can transform $L(\theta)+a\cos(\frac{\pi}{3}-\theta)$ into the form

$$-\frac{2}{\sqrt{3}}\sin(\frac{2\pi}{3}-\theta)\int_{\frac{\pi}{3}}^{\theta}\rho(\varphi)\sin(\varphi-\frac{\pi}{3})d\varphi$$
$$-\frac{2}{\sqrt{3}}\sin(\theta-\frac{\pi}{3})\int_{\theta}^{\frac{2\pi}{3}}\rho(\varphi)\sin(\frac{2\pi}{3}-\varphi)d\varphi$$
$$+\frac{2a}{\sqrt{3}}\sin\theta-a\,.$$

If we observe that $\rho(\varphi)\leqq a$, we have

$$L(\theta)+a\cos(\frac{\pi}{3}-\theta)>0 \quad \text{for} \quad \frac{\pi}{3}<\theta<\frac{2\pi}{3}\,.$$

Next consider the case $0<\theta<\frac{\pi}{3}$.

Since $L(\theta)+a$ depends on the curve of constant breadth C represented by $\rho=\rho(\varphi)$, we denote it by $F(\theta,\rho(\varphi))$ or $F(\theta,C)$.

If we denote by C' the oval $\rho=\rho(\pi+\varphi)$, which is identical with C, but rotated through the angle π, we have

$$F(\theta,C)+F(\theta,C')=F(\theta,\rho(\varphi))+F(\theta,\rho(\varphi+\pi))$$
$$=a(1-\frac{2}{\sqrt{3}}\sin(\frac{2\pi}{3}-\theta))<0\,,$$

for

$$\rho(\varphi)+\rho(\varphi+\pi)=a\,.$$

Therefore at least one of $F(\theta,C)$, $F(\theta,C')$ must be <0, for $0<\theta<\frac{\pi}{3}$.

Finally in the case $\frac{2\pi}{3}<\theta<\pi$, we can transform $L(\theta)+a(1+\cos\theta)$ into the form $F(\Theta,\rho(\psi))$ by putting $\pi-\theta=\Theta$, $\psi=\pi-\varphi$.

If we observe that for $0<\theta<\frac{\pi}{3}$

$$F(\theta,\rho(\varphi))+F(\theta,\rho(\pi+\varphi))$$

and

$$F(\theta,\rho(\psi))+F(\theta,\rho(\pi+\psi))$$

are both equal to $a(1-\frac{2}{\sqrt{3}}\sin(\frac{2\pi}{3}-\theta))$,

consequently $G=F(\theta,\rho(\varphi))-F(\theta,\rho(\pi-\varphi))$

is equal to $-\{F(\theta,\rho(\pi+\varphi))-F(\theta,\rho(\pi-(\pi+\varphi)))\}$, that is, G changes its sign when φ is changed into $\varphi+\pi$, we can conclude the existence of a constant α $(0<\alpha<\pi)$ such that

$$F(\theta, \rho(\varphi+\alpha))-F(\theta, \rho(\pi-\varphi-\alpha))=0.$$

In this case, it is also true that at least one of $F(\theta, \rho(\varphi+\alpha))$, $F(\theta, \rho(\varphi+\alpha+\pi))<0$ for $0<\theta<\frac{\pi}{3}$.

Assume for example $F(\theta, \rho(\varphi+\alpha))<0$.

Then $F(\theta, \rho(\pi-\varphi-\alpha))$ is also <0 for $0<\theta<\frac{\pi}{3}$,

or $L(\theta)+a(1+\cos\theta)<0$ for $\frac{2\pi}{3}<\theta<\pi$.

Thus, by bringing the given oval into the position $\rho=\rho(\varphi+\alpha)$ by rotating through the angle α, and comparing this with the Reuleaux triangle, we have the relation

$$L(\theta)+a<0 \qquad \text{for} \qquad 0<\theta<\frac{\pi}{3},$$

$$L(\theta)+a\cos\left(\frac{\pi}{3}-\theta\right)>0 \qquad \text{for} \qquad \frac{\pi}{3}<\theta<\frac{2\pi}{3},$$

$$L(\theta)+a(1+\cos\theta)<0 \qquad \text{for} \qquad \frac{2\pi}{3}<\theta<\pi.$$