PAPERS COMMUNICATED

98. *An Extension of the Lebesgue Measure.*

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The domain of the definition of a completely additive set function $\mu(E)$ must be a closed family (σ-Körper) of sets, which I denote by $\mathfrak{F}(\mu)$. The Lebesgue measure $m(E)$ has for its domain of the definition, $\mathfrak{F}(m)$, the closed family of sets, which are measurable in Lebesgue sense. Now there is a problem: Is there any completely additive set function $\mu(E)$, whose domain of the definition $\mathfrak{F}(\mu)$ contains $\mathfrak{F}(m)$, and the value of $\mu(E)$ at any set belonging to $\mathfrak{F}(m)$, is equal to its Lebesgue measure? In this paper, I will construct such a set function $\mu(E)$.

By the Carathéodory's theory of measure,[1] $\mu^{※}(E)$ being a measure function, if a set A satisfies the following relation

$$\mu^{※}(W)=\mu^{※}(AW)+\mu^{※}(W-AW) \tag{1}$$

for any set W of finite $\mu^{※}$-measure, then A is said to be $\mu^{※}$-measurable, and the aggregate of all such $\mu^{※}$-measurable set being a closed family $\mathfrak{F}(\mu)$, the set function $\mu(E)$ defined in $\mathfrak{F}(\mu)$ such that

$$\mu(A)=\mu^{※}(A),$$

is completely additive in $\mathfrak{F}(\mu)$.

Now let $m^{※}(E)$ be the exterior Lebesgue measure, and consider the set function

$$\mu^{※}(E)=m^{※}(E\Omega), \tag{2}$$

where Ω is the non-measurable set, constructed in the Carathéodory's treatise,[2] which has the whole space as its same-measure cover, that is, if M be any $m^{※}$-measurable set of finite $m^{※}$-measure, then

$$m(M)=m^{※}(M\Omega). \tag{3}$$

Then $\mu^{※}(E)$ is also a measure function,[3] and we have a completely

1) Carathéodory, *Vorlesungen über reelle Functionen*, zweite Aufl. (1927), 246.
2) *Ibid.*, 354.
3) *Ibid.*, 240.

additive set function $\mu(E)$ defined in the closed family $\mathfrak{F}(\mu)$ of μ※-measurable sets.

Let M be any set belonging to $\mathfrak{F}(m)$, then M belongs also to $\mathfrak{F}(\mu)$, and

$$\mu(M)=m(M). \tag{4}$$

To prove this, let W be any set of finite μ※-measure, then $W\Omega$ is of finite m※-measure, therefore, since M is m※-measurable, we have

$$m^{※}(W\Omega)=m^{※}(W\Omega M)+m^{※}(W\Omega-W\Omega M),$$

but by (2) this becomes

$$\mu^{※}(W)=\mu^{※}(WM)+\mu^{※}(W-WM),$$

therefore, M is also μ※-measurable.

When $m(M)$ is finite, we have by (2) and (3)

$$\mu(M)=m^{※}(M\Omega)=m(M).$$

But, when $m(M)$ is infinite, there exists a sequence of m※-measurable sets of finite m※-measure

$$M_1\subset M_2\subset\cdots\cdots\subset M_i\subset\cdots\cdots,$$

all of which are contained in M, and

$$\lim_{i\to\infty} m(M_i)=+\infty.$$

Then
$$\mu(M)\geqq\mu(M_i)=m(M_i)$$

for any value of i, therefore, we have

$$\mu(M)=+\infty.$$

$\mathfrak{F}(\mu)$ contains sets which do not belong to $\mathfrak{F}(m)$.

For let A be any set which does not belong to $\mathfrak{F}(m)$, and satisfies the relation

$$A\Omega=0,\text{[1]}$$

then by (2)

$$\mu^{※}(A)=0,$$

therefore, A being μ※-measurable,[2] it belongs to $\mathfrak{F}(\mu)$.

Thus, I have the required completely additive set function $\mu(E)$.

The set function $\mu^{※}(E)$ has the following property

$$m_{※}(E)\leqq\mu^{※}(E)\leqq m^{※}(E)$$

for any set E.

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1) For example, let A be the complementary set of Ω.

2) H. Hahn: *Theorie der reellen Funktionen*, **1** (1921), 429.

For, first we have

$$\mu^{※}(E)=m^{※}(E\Omega)\leqq m^{※}(E)\,.$$

Next let $\underline{E}$ be the same-measure nucleus of E, that is

$$\underline{E}\subseteqq E \qquad \text{and} \qquad m(\underline{E})=m_{※}(E)\,,$$

then by (4)

$$m_{※}(E)=\mu(\underline{E})\leqq \mu^{※}(E)\,.$$