PAPERS COMMUNICATED

37. *The Foundation of the Theory of Displacements, III.*

(Application to a manifold of matrices.)

By Akitsugu KAWAGUCHI.
Mathematical Institute, Hokkaido Imperial University, Sapporo.

(Comm. by M. FUJIWARA, M.I.A., Feb. 12, 1934.)

Three kinds of displacements for a manifold of matrices are considered in this paper from the standpoint of the general theory set out in my previous paper (F.D.I.) and also from that of its application to a manifold with a linear connection.

1. Let us consider a manifold of finite dimensions M with a coordinate system x^λ ($\lambda=1, 2, \ldots\ldots, n$) and associate a manifold of matrices to its each point, where under the underlying isomorphism between any two manifolds $\overline{M}$ in § 6 (F.D.I.) the corresponding matrices have corresponding elements of the same values. Then (10) in F.D.I. becomes

$$\nabla A = dA + \Gamma(A) \tag{1}$$

for a matrix A in $\overline{M}$ determined uniquely for every point of M, where $\Gamma(A)$ is a matrix depending on x^λ and the differential dA is a matrix, whose elements are differentials of that of A. Our object is not to consider such a general displacement, but a special one such that

$$\nabla A = dA + \Gamma A + A\Gamma', \tag{2}$$ [1]

where Γ and Γ' are matrices independent of A. This displacement is clearly linear and has many interesting properties, as we see in the following. When Γ and Γ' are linear forms with respect to dx^λ and have such forms that $\Gamma=\Gamma_\lambda(x)dx^\lambda$, $\Gamma'=\Gamma'_\lambda(x)dx^\lambda$, then it follows from (2)

$$\nabla_\lambda A = \frac{\partial A}{\partial x^\lambda} + \Gamma_\lambda A + A\Gamma'_\lambda . \tag{3}$$

2. For the covariant derivatives of the inverse matrix A^{-1} of A it seems to be most natural to define them in the same manner as

---

1) We may also define such a differentiation, that $\nabla a_{ij} = da_{ij} + \sum_{k,l} \Gamma^{kl}_{ij} a_{kl}$ where $A=((a_{ij}))$. A special case of this connection has been studied by S. Hokari: Über die Bivektorübertragung, Journal of the Faculty of Science, Hokkaido Imperial University, Series I, **2** (1934), 103-117.

those of A, as A^{-1} is also a matrix with the same property as A. Neverthless we may define them independently from (3)

$$\nabla_\lambda A^{-1}=\frac{\partial A^{-1}}{\partial x^\lambda}+\overset{(-)}{\Gamma}_\lambda A^{-1}+A^{-1}\overset{(-)}{\Gamma}{}'_\lambda\,, \tag{4}$$

which is more general than the previous one. We look upon A and A^{-1} as not of the same quality, and put a matrix into one of two categories K, K^{-1} according as it is considered as A or A^{-1}. This concept is similar to that of orientation. From this standpoint we consider one more category K^{+1}, and define for any A^{+1} belonging to this category

$$\nabla_\lambda A^{+1}=\frac{\partial A^{+1}}{\partial x^\lambda}+\overset{(+)}{\Gamma}_\lambda A^{+1}+A^{+1}\overset{(+)}{\Gamma}{}'_\lambda\,, \tag{5}$$

where $A^{+1}A^{-1}=U$, U being the unit matrix, and more generally $A^{+1}B^{-1}=C$.[1)]

3. From $A^{+1}B^{-1}=C$, we have

$$\begin{aligned}\nabla_\lambda C=\nabla_\lambda(A^{+1}B^{-1})&=(\nabla_\lambda A^{+1})B^{-1}+A^{+1}(\nabla_\lambda B^{-1})\\ &+A^{+1}(\overset{(+)}{\Gamma}{}'_\lambda+\overset{(-)}{\Gamma}_\lambda)B^{-1}+(\Gamma_\lambda-\overset{(+)}{\Gamma}_\lambda)A^{+1}B^{-1}+A^{+1}B^{-1}(\Gamma'_\lambda-\overset{(-)}{\Gamma}{}'_\lambda)\,.\end{aligned} \tag{6}$$

As the product matrix C of two matrices A, B, belonging to K, belongs also to K, we soon obtain from (3)

$$\nabla_\lambda C=(\nabla_\lambda A)B+A(\nabla_\lambda B)-A\Lambda_\lambda B\,, \tag{7}$$

where $\Lambda_\lambda=\Gamma_\lambda+\Gamma'_\lambda$. For the unit matrix U (6) and (7) become

$$\begin{aligned}\nabla_\lambda U&=-\Lambda_\lambda\\ &=(\nabla_\lambda A^{+1})A^{-1}+A^{+1}(\nabla_\lambda A^{-1})-A^{+1}(\overset{(+)}{\Gamma}{}'_\lambda+\overset{(-)}{\Gamma}_\lambda)A^{-1}\\ &\qquad+(\Gamma_\lambda-\overset{(+)}{\Gamma}_\lambda)+(\Gamma'_\lambda-\overset{(-)}{\Gamma}{}'_\lambda)\,,\end{aligned} \tag{8}$$

for $U=U^2=A^{+1}A^{-1}$. When we put $C=U=AB$, it follows

$$\nabla_\lambda B=-B(\nabla_\lambda A)B+B\Lambda_\lambda+\Lambda_\lambda B\,. \tag{9}$$

We call the one-parameter system of matrices the *parallel system* in M, which is a solution of the differential equations $\nabla_\lambda A\dfrac{dx^\lambda}{dt}=0$. This is analogous to the parallel vector system in an ordinary manifold with a linear connection. The conditions of integrability of the differential equations $\nabla_\lambda A=0$ are

$$2\nabla_{[\mu}\nabla_{\lambda]}A\equiv K_{\lambda\mu}A+AK'_{\lambda\mu}=0\,, \tag{10}$$

---

1) Contrast the concept of these three categories with that of $|1|$, $+1$, -1 or v^λ_μ, $v^{\lambda\mu}$, $v_{\lambda\mu}$.

where

$$(11)\qquad \frac{1}{2}K_{\lambda\mu}=\frac{\partial\Gamma_{[\lambda}}{\partial x^{\mu]}}+\Gamma_{[\mu}\Gamma_{\lambda]}\,,\quad \frac{1}{2}K'_{\lambda\mu}=\frac{\partial\Gamma'_{[\lambda}}{\partial x^{\mu]}}+\Gamma'_{[\mu}\Gamma'_{\lambda]}$$

are called the *curvature matrices*, and we have also similar conditions for $\nabla_\lambda A^{\pm1}=0$. In order that for any matrix A (10) should hold, it must be $K_{\lambda\mu}=K'_{\lambda\mu}=0$, which is the case where the manifold is flat. We can obtain the following identities for the curvature matrices, which are analogous to the so-called Bianchi's identities,

$$(12)\qquad \nabla_{[\lambda}K_{\nu\mu]}-K_{[\nu\mu}\Lambda_{\lambda]}=0\,.$$

4. The parameters and the curvature matrices are transformed by any change of coordinates x^λ in a similar manner to ordinary vectors and tensors respectively, for example,

$$(13)\qquad \overline{\Gamma}_\mu=\frac{\partial x^\nu}{\partial\overline{x}^\mu}\Gamma_\nu\,,\quad \overline{K}_{\lambda\mu}=K_{\nu\omega}\frac{\partial x^\nu}{\partial\overline{x}^\lambda}\frac{\partial x^\omega}{\partial\overline{x}^\mu}\,,$$

but $\nabla_\mu\nabla_\lambda A$ not to ordinary tensor and

$$(14)\qquad \overline{\nabla}_\mu\overline{\nabla}_\lambda A=\nabla_\nu\nabla_\omega A\frac{\partial x^\nu}{\partial\overline{x}^\mu}\frac{\partial x^\omega}{\partial\overline{x}^\lambda}+\frac{\partial^2 x^\nu}{\partial\overline{x}^\mu\partial\overline{x}^\lambda}\nabla_\nu A\,.$$

If we intend to define the second covariant derivatives as that they are transformed in the same manner as the curvature matrices by any change of x^λ, it is sufficient to introduce the functions $\gamma^\nu_{\lambda\mu}$ having the property of parameters of an affine connection[1] and to put

$$(15)\qquad \overset{*}{\nabla}_\mu\nabla_\lambda A=\nabla_\mu\nabla_\lambda A-\gamma^\nu_{\lambda\mu}\nabla_\nu A\,.$$

This connection lies also in the sphere of application of the general theory. In fact, $\overline{M}$ may be regarded here as having a matrix A and a number of matrices $\nabla_\lambda A$ as its elements.

5. Let $\overline{A}=VAW$ be the fundamental transformation in $\overline{M}$, where V and W are both matrices and $WV=U$. We then have the change of Γ_λ and Γ'_λ due to this transformation

$$(16)\qquad \overline{\Gamma}_\lambda=V\Gamma_\lambda W-\frac{\partial V}{\partial x^\lambda}W\,,\quad \overline{\Gamma}'_\lambda=V\Gamma'_\lambda W-V\frac{\partial W}{\partial x^\lambda}\,.$$

These are the necessary and sufficient conditions for $\nabla(VAW)=V\nabla AW$, A being an arbitrary matrix.

6. The parameter matrices Γ_λ, Γ'_λ may be regarded as both square matrices and they depend in general on the numbers of rows p and columns q of the matrix A, to which the covariant differentiation be

1) Naturally a general linear connection can stand for the affine connection.

applied, and the order of Γ_λ is equal to q and that of Γ'_λ to p. For this reason we may denote them by $\overset{pq}{\Gamma}_\lambda$, $\overset{pq}{\Gamma}{}'_\lambda$ respectively, where $p, q=1, 2, \ldots\ldots$, i.e. we have an infinitely large number of parameters. This fact means that the number of dimensions of $\overline{M}$ may be regarded as being indeterminate.

7. Now we shall apply the above-mentioned theory to a manifold with a linear connection $\gamma^{\nu}_{\lambda\mu}$. Let A be an ennuple $\underset{\lambda}{e}{}^{\mu}$ and $\Gamma_\lambda=((\gamma^{\nu}_{\mu\lambda}))$, $\Gamma'_\lambda=0$, then the parallel system of matrices are nothing but the system of geodesically moving ennuples along a curve[1] and we have $K_{\lambda\mu}=((R^{\cdot\cdot\nu}_{\omega\lambda\mu}))$. When A is regarded as a linear transformation $\bar{v}^{\nu}=A^{\nu}_{\lambda}v^{\lambda}$ in the ordinary vector space, and $\Gamma_\lambda=-\Gamma'_\lambda=((\gamma^{\nu}_{\mu\lambda}))$, then we obtain the theory of linear transformation groups, which is of course a special case of Cartan's theory of transformation groups.[2] (12) coincide perfectly with Bianchi's identities. We can moreover in this case let A^{+1} and A^{-1} correspond to a contra- and covariant tensor $A^{\lambda\mu}$, $A_{\lambda\mu}$ respectively; then we know that Γ_λ, Γ'_λ proceed to the parameters of connection of a tensor manifold.

8. We can better generalize our theory by taking M also as a manifold of matrices. Then the covariant derivatives of a matrix A form a matrix A_X, whose elements are matrices, and there are two kinds of the second covariant derivatives. We have many other interesting facts, but their detailed theory will be left to the next paper.

---

1) See on the moving ennuples, E. Laura: La teoria delle matrici e il metodo dell'n. edro mobile, Rendiconti d. Seminario Mat. d. R. Univ. di Padova, **1** (1930), 85-109.

2) E. Cartan: La géométrie des groupes de transformations, Journal de Math., sér. 9, **6** (1927), 1-119, etc.