PAPERS COMMUNICATED

108. On the Wiener's Formula.

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1. Wiener[1] has proved that:

If
$$\mathfrak{M}\{f\}=\lim_{x\to\infty}\frac{1}{x}\int_0^x f(\xi)d\xi$$
exists and is finite and $\frac{1}{x}\int_0^x |f(\xi)|d\xi$ is bounded in $(0, \infty)$, then
$$\lim_{\varepsilon\to 0}\frac{1}{\pi}\int_0^\infty f(x)\frac{\sin^2\varepsilon x}{\varepsilon x^2}dx=\mathfrak{M}\{f\}. \tag{1}$$

Bochner[2] has replaced the kernel $\frac{\sin^2 x}{x^2}$ in (1) by a general function $K(x)$ and found the conditions for the validity of
$$\lim_{\varepsilon\to 0}\int_0^\infty f\left(\frac{x}{\varepsilon}\right)K(x)dx=\mathfrak{M}\{f\}\int_0^\infty K(x)dx. \tag{2}$$
Bochner named (2) the Wiener's formula.

In this paper, we treat the conditions of validity of (2).

2. *Theorem 1.* Suppose that (i) $K(x)$ is absolutely continuous in any finite interval, (ii) $K(x)$ is absolutely integrable in $(0, \infty)$, (iii) $xK(x)$ is of bounded variation in $(0, \infty)$, and (iv) $\frac{1}{x}\int_0^x f(\xi)d\xi$ is bounded in $(0, \infty)$, and (v) the limit $\mathfrak{M}\{f\}=\lim_{x\to\infty}\frac{1}{x}\int_0^x f(\xi)d\xi$ exists and is finite. Then we have
$$\lim_{\varepsilon\to 0}\int_0^\infty f\left(\frac{x}{\varepsilon}\right)K(x)dx=\mathfrak{M}\{f\}\int_0^\infty K(x)dx. \tag{2}$$

Proof. Without loss of generality, we can suppose that

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1) Wiener, Math. Zeits., **24** (1926); ——, Journ. Math. and Phys. M. I. T., **5** (1926); ——, Journ. London Math. Soc., **2** (1927). *Cf.* Bochner-Hardy, Journ. London Math. Soc., **1** (1926); Jacob, Journ. London Math. Soc., **3** (1928); Littauer, Journ. London Math. Soc., **4** (1929); Wiener, Acta Math., **55** (1931).

2) Bochner, Vorlesungen über Fouriersche Integrale, 1933, pp. 30–32.

$$\mathfrak{M}\{f\}=0\,. \tag{3}$$

Then it is sufficient to prove

$$\lim_{\varepsilon\to 0}\int_0^\infty f\left(\frac{x}{\varepsilon}\right)K(x)dx=0 \tag{2'}$$

instead of (2). And we can suppose that $\varepsilon<1$.

If we put $F(x)=\int_0^x f(\xi)d\xi$, we have, by integration by parts and the condition (i),

$$\int_A^B f\left(\frac{x}{\varepsilon}\right)K(x)dx=\left[K(x)\int_0^x f\left(\frac{\xi}{\varepsilon}\right)d\xi\right]_{x=A}^B-\int_A^B K'(x)dx\int_0^x f\left(\frac{\xi}{\varepsilon}\right)d\xi$$

$$=K(B)\varepsilon F\left(\frac{B}{\varepsilon}\right)-K(A)\varepsilon F\left(\frac{A}{\varepsilon}\right)-\varepsilon\int_A^B K'(x)F\left(\frac{x}{\varepsilon}\right)dx$$

$$=BK(B)\frac{F\left(\frac{B}{\varepsilon}\right)}{\frac{B}{\varepsilon}}-AK(A)\frac{F\left(\frac{A}{\varepsilon}\right)}{\frac{A}{\varepsilon}}-\varepsilon\int_A^B K'(x)F\left(\frac{x}{\varepsilon}\right)dx\,.$$

By the condition (iii), $xK(x)$ is bounded in $(0,\infty)$, that is, there is a constant M such that $|xK(x)|<M$. By (3), for any $\eta(>0)$, we can find A_0 such that $\left|\frac{F(x)}{x}\right|<\eta$ for $x>A_0$. Then we have

$$\left|\int_A^B f\left(\frac{x}{\varepsilon}\right)K(x)dx\right|\leqq 2\eta M+\eta\int_A^B|xK'(x)|\,dx \tag{4}$$

for $A>A_0$. By the identity

$$\frac{d}{dx}\{xK(x)\}=K(x)+xK'(x)$$

and the conditions (i), (ii) and (iii), $xK'(x)$ is absolutely integrable in $(0,\infty)$, then there is an N such that $\int_0^\infty|xK'(x)|\,dx<N$. Therefore, letting $B\to\infty$ in (4), we have

$$\left|\int_A^\infty f\left(\frac{x}{\varepsilon}\right)K(x)dx\right|\leqq(2M+N)\eta \tag{5}$$

for an $A>A_0$.

We have, by (i) and integation by parts,

$$\int_0^A f\left(\frac{x}{\varepsilon}\right)K(x)dx=K(A)\varepsilon F\left(\frac{A}{\varepsilon}\right)-\varepsilon\int_0^A K'(x)F\left(\frac{x}{\varepsilon}\right)dx\,. \tag{6}$$

If we take a such that $\int_0^a |xK'(x)|\,dx < \eta$, and L such that $\left|\frac{F(x)}{x}\right| < L$ for x in $(0, \infty)$, then

$$\left|\varepsilon\int_0^a K'(x)F\left(\frac{x}{\varepsilon}\right)dx\right| \leqq L\int_0^a |xK'(x)|\,dx < \eta L\,, \tag{7}$$

and there is an ε_0 such that $\left|\frac{\varepsilon}{x}F\left(\frac{x}{\varepsilon}\right)\right| < \eta$ for any $x > a$ and any $\varepsilon < \varepsilon_0$. Hence

$$\begin{aligned}\left|\varepsilon\int_a^A K'(x)F\left(\frac{x}{\varepsilon}\right)dx\right| &\leqq \int_a^A \left|xK'(x)\right|\cdot\left|\frac{\varepsilon}{x}F\left(\frac{x}{\varepsilon}\right)\right|dx \\ &\leqq \eta\int_a^A |xK'(x)|\,dx \\ &\leqq \eta\int_0^\infty |xK'(x)|\,dx \\ &\leqq \eta N.\end{aligned} \tag{8}$$

We have, by (6), (7) and (8),

$$\left|\int_0^A f\left(\frac{x}{\varepsilon}\right)K(x)dx\right| \leqq (M+L+N)\eta$$

for $\varepsilon \leqq \varepsilon_0$. Thus (2′) is proved.

3. Similarly we can prove the following theorem.

Theorem 2. Suppose that (i) $K(x)$ is absolutely continuous in any finite interval and there is a function $\bar{K}(x)$ such that (ii) $|K(x)| \leqq \bar{K}(x)$, (iii) $\bar{K}(x)$ is absolutely continuous in any finite interval and absolutely integrable in $(0, \infty)$, (iv) $x\bar{K}(x)$ is of bounded variation in $(0, \infty)$ and tends to zero as $x \to \infty$. Further suppose that (v) $\frac{1}{x}\int_0^x |f(\xi)|\,d\xi$ is bounded in $(0, \infty)$, and (vi) the limit

$$\mathfrak{M}\{f\} = \lim_{x\to\infty}\frac{1}{x}\int_0^x f(\xi)d\xi$$

exists and is finite. Then we have

$$\lim_{\varepsilon\to 0}\int_0^\infty f\left(\frac{x}{\varepsilon}\right)K(x)dx = \mathfrak{M}\{f\}\int_0^\infty K(x)dx\,.$$

This is a generalization of the Bochner's theorem.

4. Further we can prove the following theorem.

Theorem 3. Let k be a positive integer. Suppose that (i) $K(x)$ is absolutely integrable in $(0, \infty)$, (ii) $K^{(k-1)}(x)$ is absolutely continuous in any finite interval, (iii) $x^i\mathrm{K}^{(i-1)}(x)$ $(i=1, 2, \ldots\ldots, k)$ is of bounded variation in $(0, \infty)$, and (iv) $x^i K^{(i-1)}(x)$ $(i=1, 2, \ldots\ldots, k-1)$ tends to zero as $x \to \infty$. Further suppose that (v) $\frac{1}{x}\int_0^x f(\xi)d\xi$ is bounded in $(0, \infty)$, and (vi)

$$\mathfrak{M}_k\{f\} = \lim_{x_1 \to \infty} \frac{1}{x_1}\int_0^{x_1} \frac{dx_2}{x_2}\int_0^{x_2} \ldots\ldots \frac{dx_k}{x_k}\int_0^{x_k} f(\xi)d\xi$$

exists and is finite. Then we have

$$\lim_{\varepsilon \to 0}\int_0^\infty f\left(\frac{x}{\varepsilon}\right)K(x)dx = \mathfrak{M}_k\{f\}\int_0^\infty K(x)dx .$$