# 153. *Note on a Certain Multivalent Function.*

By Tetuzi ITIHARA.
Second Higher School, Sendai.

(Comm. by T. YOSIE, M.I.A., Nov. 12, 1934.)

In this note we prove a theorem on a certain multivalent function.

*Theorem.* *Let*

$$w=f(z)=\frac{1}{z^k}+a_kz^k+a_{k+1}z^{k+1}+a_{k+2}z^{k+2}+\cdots\cdots+a_nz^n+\cdots\cdots \qquad (a_k\neq 0)$$

*be regular and k-valent in* $0<|z|<1$, *then*

$$k|a_k|^2+(k+1)|a_{k+1}|^2+(k+2)|a_{k+2}|^2+\cdots\cdots+n|a_n|^2+\cdots\cdots\leqq k\,.$$

Proof. We consider at first a circle $|z|=r$ $(0<r<1)$, then we may write

$$\Big|\sum_{n=k+1}^{\infty}a_nz^{n-k}\Big|<\delta\,,$$

where $\delta$ denotes a certain positive constant. Therefore, if we write

$$\zeta=\frac{1}{z^k}+a_kz^k$$

$$|w-\zeta|<\delta|z|^k=\delta r^k$$

and so

$$|w-2\sqrt{a_k}|+|w+2\sqrt{a_k}|\leqq 2|w-\zeta|+|\zeta-2\sqrt{a_k}|+|\zeta+2\sqrt{a_k}|$$

$$<2\delta r^k+|\zeta-2\sqrt{a_k}|+|\zeta+2\sqrt{a_k}|\,.$$

Now, since

$$|\zeta-2\sqrt{a_k}|+|\zeta+2\sqrt{a_k}|=|z^{\frac{k}{2}}-a_k^{\frac{1}{2}}z^{\frac{k}{2}}|^2+|z^{-\frac{k}{2}}+a_k^{\frac{1}{2}}z^{\frac{k}{2}}|^2$$

$$=2\{|z^{-\frac{k}{2}}|^2+|a_k^{\frac{1}{2}}z^{\frac{k}{2}}|^2\}$$

$$=2\{\frac{1}{r^k}+|a_k|r^k\}\,,$$

it follows that

$$|w-2\sqrt{a_k}|+|w+2\sqrt{a_k}|<2\left\{\frac{1}{r^k}+(|a_k|+\delta)r^k\right\}. \qquad (1)$$

Thus the image of $|z|=r$ by $w=f(z)$ lies in the elliptic domain (1) on the $w$-plane. Let $A$ denote its area, then

$$A=\pi\left\{\frac{1}{r^k}+(|a_k|+\delta)r^k\right\}\sqrt{\left\{\frac{1}{r^k}+(|a_k|+\delta)r^k\right\}^2-4|a_k|}$$

$$=\pi\left\{\frac{1}{r^{2k}}-(|a_k|+\delta)^2 r^{2k}\right\}\sqrt{1+\frac{4\delta}{Q^2}}\ ,$$

where

$$Q=\frac{1}{r^k}-(|a_k|+\delta)r^k\,.$$

It is easy to see that $Q>\frac{1}{2r^k}$ by taking $|a_k|>\delta$ and

$$0<r<\sqrt[2k]{\frac{1}{4|a_k|}}\ ,$$

and so

$$A<\pi\left\{\frac{1}{r^{2k}}-(|a_k|+\delta)^2 r^{2k}\right\}\sqrt{1+16\delta r^{2k}}$$

$$<\pi\left\{\frac{1}{r^{2k}}-(|a_k|+\delta)^2 r^{2k}\right\}(1+8\delta r^{2k})$$

$$<\frac{\pi}{r^{2k}}(1+8\delta r^{2k})=\pi\left(\frac{1}{r^{2k}}+8\delta\right).$$

We shall now assume that $0<r<R<1$, then the image of $|z|=R$ by $f(z)$ on the $w$-plane lies within the elliptic domain (1), since semiaxis of (1) increases without limit as $r\to 0$. Therefore, the image of $r\leqq|z|\leqq R$ lies also within (1), and has the area

$$\pi\left\{k\left(\frac{1}{r^{2k}}-\frac{1}{R^{2k}}\right)+\sum_{n=k}^{\infty}n|a_n|^2(R^{2n}-r^{2n})\right\}.$$

It follows from the $k$-valency of $f(z)$ in $0<|z|<1$,

$$\pi\left\{k\left(\frac{1}{r^{2k}}-\frac{1}{R^{2k}}\right)+\sum_{n=k}^{\infty}n|a_n|^2(R^{2n}-r^{2n})\right\}<kA<k\pi\left(\frac{1}{r^{2k}}+8\delta\right),$$

$$\therefore\quad -\frac{k}{R^{2k}}+\sum_{n=k}^{\infty}n|a_n|^2(R^{2n}-r^{2n})<8k\delta\,.$$

Finally, by $r\to 0$, we have

$$\sum_{n=k}^{\infty}n|a_n|^2R^{2n}\leqq\frac{k}{R^{2n}}+8k\delta\,,$$

and by $R\to 1$, $\delta\to 0$

$$\sum_{n=k}^{\infty}n|a_n|^2\leqq k\,.$$

This proves the theorem.