PAPERS COMMUNICATED

151. *Some Remarks on a Theorem Concerning Star-shaped Representation of an Analytic Function.*

By Kenjirô NABETANI.

Mathematical Institute, Tohoku Imperial University, Sendai.

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1. The following theorem is originally due to Mr. S. Takahashi[1] and was completed by Mr. K. Noshiro.[2]

Let $f(z)=z+a_2z^2+\cdots\cdots+a_nz^n+\cdots\cdots$ [3]

be regular for $|z|<1$ and

$$(1)\qquad |f'(z)|<M \qquad (M>1)$$

for $|z|<1$, then the circle $|z|<\frac{1}{M}$ is represented conformally by $f(z)$ on a star-shaped domain with respect to the origin. Moreover the limiting case is attained by the function

$$f(z)=M\int_0^z\frac{1-Mz}{M-z}dz=M\left(Mz+(M^2-1)\log\left(1-\frac{z}{M}\right)\right),$$

whose derived function has a zero-point at $z=\frac{1}{M}$.

Examining the proof of Mr. Noshiro for this theorem, we see that the condition (1) can evidently be replaced by

$$(2)\qquad \left\{\frac{1}{2\pi}\int_0^{2\pi}|f'(re^{i\theta})|^2d\theta\right\}^{\frac{1}{2}}\leqq M \quad \text{for} \quad 0\leqq r<1\,.$$

2. Now we can naturally propose the following problems.

(I) To find the radius of the greatest circle with the center at the origin which is represented conformally by every $f(z)$ on a star-shaped domain with respect to the origin under the condition

$$(3)\qquad \left\{\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^2d\theta\right\}^{\frac{1}{2}}<M \quad (M>1) \quad \text{for} \quad 0\leqq r<1\,.$$

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1) Tohoku Math. Journ. **33** (1931), p. 58.

2) Proc. **7** (1932).

3) In this note, $f(z)$ means throughout the function of this form.

(II) To find the radius of the same property as in (I) under the condition

$$(4) \qquad |f(z)|<M \quad (M>1) \quad \text{for} \quad |z|<1.$$

This problem has been solved by Mr. Dieudonné,[1] and the required value of the radius is $M-\sqrt{M^2-1}$.

(III) To find the radius of the greatest circle with center at the origin which is represented conformally by every $f(z)$ on a convex domain under the condition (3).

(IV) To solve the same problem as (III) under the condition (4).

(V) To solve the same problem as (III) under the condition (1).

Among these problems, (V) and (IV) are equivalent to (I) and (II) respectively which can be seen from the following

Lemma 1. The circle $|z|<r$ is represented conformally by $f(z)$ on a star-shaped domain, when and only when $F(z)$, defined by $f(z)=zF'(z)$, represents $|z|<r$ on a convex domain.

In the next lines I will solve the problems (I) and (III) by the same method as Mr. Noshiro.

3. First, the required value in the problem (I) is given by the positive root of

$$\frac{R^2}{(1-R^2)^3}=\frac{1}{M^2-1}.$$

To prove this, we make use of a lemma[2] given by Mr. Noshiro.

Lemma 2. Let $f(z)$ be regular in $|z|<1$. If

$$\sum_{n=2}^{\infty} n|a_n|r^{n-1}<1 \quad \text{for} \quad 0\leqq r<1,$$

the circle $|z|\leqq r$ is mapped by $f(z)$ on a star-shaped domain with respect to the origin.

Now by the condition (3) we have

$$1+|a_2|^2+|a_3|^2+\cdots\cdots=\lim_{r\to 1}\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^2 d\theta \leqq M^2,$$

consequently

$$\sum_{n=2}^{\infty} n|a_n|r^{n-1}\leqq\Big(\sum_{n=2}^{\infty}|a_n|^2\Big)^{\frac{1}{2}}\Big(\sum_{n=2}^{\infty}n^2r^{2(n-1)}\Big)^{\frac{1}{2}}\leqq\sqrt{M^2-1}\sqrt{\frac{r^2}{(1-r^2)^3}}<1$$

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1) Ann. l'École norm. sup. **48** (1931), p. 350.

2) K. Noshiro: loc. cit. We can find this also in J. W. Alexander II, Annale of Math. **17** (1915), p. 22.

for $\qquad 0 \leqq r < R \qquad$ and $\qquad \dfrac{R^2}{(1-R^2)^3} = \dfrac{1}{M^2-1}$.

Hence by the lemma our assertion is proved. If a function which satisfies the assumption of this theorem has a zero point of its derived function on the circle $|z|=R$, then it must be

$$\sum_{n=2}^{\infty} n|a_n|R^{n-1}=1.$$

In order that this holds good we must have

$$|a_n|=(M^2-1)nR^{n-1} \qquad (n=2, 3, \ldots\ldots).$$

Here we are led to consider the function

$$f(z)=z-(M^2-1)\sum_{n=2}^{\infty} nR^{n-1}z^n = M^2z - \frac{(M^2-1)z}{(1-Rz)^2}.$$

For this function we have

$$\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^2 d\theta < \frac{1}{2\pi}\int_0^{2\pi}|f(e^{i\theta})|^2 d\theta = 1+(M^2-1)\sum_{n=2}^{\infty} n^2R^{2n-2} = M^2,$$

and $\qquad f'(R)=1-(M^2-1)\sum_{n=2}^{\infty} n^2R^{n-1}z^{n-1}=0.$

Hence R can not be replaced by any greater number.

Next, the value required in the problem (III) is given by the positive root of

$$\sum_{n=1}^{\infty} n^4R^{2(n-1)} = \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{1+Re^{i\theta}}{1-Re^{i\theta}}\right|^2 d\theta = \frac{M^2}{M^2-1}.$$

Proof can easily be done by the aid of the following lemma, which is an immediate consequence of the lemmas 1 and 2.

Lemma 3. If

$$\sum_{n=2}^{\infty} n^2|a_n|r^{n-1}<1 \qquad \text{for} \qquad 0<r<1,$$

the circle $|z|\leqq r$ is mapped conformally by $f(z)$ on a convex domain.

To prove that the value of R given above is the best possible one, we take the function

$$f(z)=z-(M^2-1)\sum_{n=2}^{\infty} n^2R^{n-1}z^n = M^2z(M^2-1)z\frac{1+Rz}{(1-Rz)^2},$$

for which

$$\left\{\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^2 d\theta\right\}^{\frac{1}{2}} < \left\{\frac{1}{2\pi}\int_0^{2\pi}|f(e^{i\theta})|^2 d\theta\right\}^{\frac{1}{2}} = M$$

and

$$1+R\left\{\frac{zf''(z)}{f'(z)}\right\}^{1)}=1-R\left\{\frac{(M^2-1)\sum_{n=2}^{\infty}n^3(n-1)R^{n-1}z^{n-1}}{1-(M^2-1)\sum_{n=2}^{\infty}n^3R^{n-1}z^{n-1}}\right\}$$

$$=1-R\left\{\frac{(M^2-1)\sum_{n=2}^{\infty}n^4(Rz)^{n-1}-(M^2-1)\sum_{n=2}^{\infty}n^3(Rz)^{n-1}}{1-(M^2-1)\sum_{n=2}^{\infty}n^3(Rz)^{n-1}}\right\}<0,$$

for $|z|>R$,

so $f(z)$ does not map conformally any circle with the center at the origin and a radius greater than R on a convex domain.

4. If we take in place of $f(z)$ the polynomial

$$P_n(z)=z+a_2z^2+a_3z^3+\cdots\cdots+a_nz^n \qquad (n\geqq 2),$$

we get more precise results. For example, the radius r_n required in the problem (III) for this function is given by the positive root of

$$\sum_{\nu=1}^{\infty}\nu^4 r_n^{2(\nu-1)}=\frac{M^2}{M^2-1}.$$

We can also easily answer the problems (I), (V) and give analogous results to the theorem stated in §1 with the condition (2) for the function $P_n(z)$.

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1) cf. Bieberbach, Lehrbuch fer Funktionentheorie, II (1931), p. 81. $R(\zeta)$ denotes the real part of ζ as usual.