4. *A General Convergence Theorem.*

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1. S. Bochner[1] proved the following theorems:

Theorem 1. If $f(\xi)$ is bounded in $(-\infty, +\infty)$ and $K(\xi)$ is absolutely integrable in $(-\infty, +\infty)$, then we have

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} f\left(x+\frac{\xi}{n}\right)K(\xi)d\xi=f(x)\int_{-\infty}^{\infty}K(\xi)d\xi. \qquad (1)$$

Theorem 2. If (1°) $f(\xi)$ is absolutely integrable in $(-\infty, +\infty)$, (2°) $f(\xi)$ is continuous at $\xi=x$, (3°) $K(\xi)$ is absolutely integrable in $(-\infty, +\infty)$, (4°) $K(\xi)$ is bounded in $(-\infty, +\infty)$ and (5°) $K(\xi)=o(|\xi|^{-1})$ as $|\xi|\to\infty$, then we have (1).

In this paper the following associated theorem is proved:

Theorem 3. If (1°) $\dfrac{f(\xi)}{1+|\xi|}$ and $\dfrac{f^2(\xi)}{1+|\xi|}$ are absolutely integrable in $(-\infty, +\infty)$, (2°) $f(\xi)$ is continuous at $\xi=x$ and (3°) $K(\xi)$ and $\xi K^2(\xi)$ are absolutely integrable in $(-\infty, +\infty)$, then we have (1).

2. We begin with some lemmas.

Lemma 1. If $h(\eta)$ is absolutely integrable in $(-\infty, +\infty)$ and $h(\eta)$ tends continuously to a limit $h(-\infty)$ as $\eta\to-\infty$, then we have

$$\lim_{\nu\to\infty}\frac{1}{\pi}\int_{-\infty}^{\infty} h(\eta-\nu)\frac{\sin^2\lambda(\xi-\eta)}{\lambda(\xi-\eta)^2}d\eta=h(-\infty), \qquad (2)$$

boundedly for any ξ in $(-\infty, +\infty)$, λ being a fixed constant.

Proof. Without loss of generality, we may suppose that $h(-\infty)=0$.

$$\begin{aligned} J&=\int_{-\infty}^{\infty} h(\eta-\nu)\frac{\sin^2\lambda(\xi-\eta)}{\lambda(\xi-\eta)^2}d\eta \\ &=\int_{-\infty}^{\infty} h(\zeta)\frac{\sin^2\lambda(\xi-\zeta-\nu)}{\lambda(\xi-\zeta-\nu)^2}d\zeta \\ &=\int_{-\infty}^{A}+\int_{A}^{\infty} h(\zeta)\frac{\sin^2\lambda(\xi-\zeta-\nu)}{\lambda(\xi-\zeta-\nu)^2}d\zeta \\ &=J_1+J_2, \qquad \text{say.} \end{aligned}$$

1) S. Bochner: Fouriersche Integral, 1933. Cf. T. Takahashi and S. Izumi: Science Reports, Tohoku Univ., 1934.

For any positive number ε, there is an A such that $|h(\zeta)|<\frac{\varepsilon}{2\pi\lambda}$ for $\zeta<A$, and then

$$|J_1|\leqq\frac{\varepsilon}{2\pi\lambda}\int_{-\infty}^{A}\frac{\sin^2\lambda(\xi-\zeta-\nu)}{\lambda(\xi-\zeta-\nu)^2}d\zeta$$
$$<\frac{\varepsilon}{2\pi\lambda}\int_{-\infty}^{\infty}\frac{\sin^2\lambda\zeta'}{\lambda\zeta'^2}d\zeta'=\frac{\varepsilon}{2}.$$

As $h(\zeta)$ is absolutely integrable, there is an integer ν_0, such that

$$|J_2|\leqq\underset{A\leqq\zeta<\infty}{\mathrm{Max}}\left|\frac{\sin^2\lambda(\xi-\zeta-\nu)}{\lambda(\xi-\zeta-\nu)^2}\right|\cdot\int_A^{\infty}|h(\zeta)|d\zeta\leqq\frac{\varepsilon}{2}$$

for $\nu\geqq\nu_0$. Hence

$$|J|<\varepsilon$$

for $\nu\geqq\nu_0$. Thus we get (2).

Lemma 2. If $K^*(\xi)$ is squarely integrable in $(-\infty, +\infty)$ and we put

$$K_\lambda^*(\eta)=\frac{1}{\pi}\int_{-\infty}^{\infty}K^*(\xi)\frac{\sin^2\lambda(\xi-\eta)}{\lambda(\xi-\eta)^2}d\xi,$$

then we have

$$\lim_{\lambda\to\infty}\int_{-\infty}^{\infty}|K^*(\eta)-K_\lambda^*(\eta)|^2d\eta=0. \tag{3}$$

Proof.[1] Let $k^*(\xi)$ and $k_\lambda^*(\xi)$ be the Fourier transform of $K^*(\xi)$ and $K_\lambda^*(\xi)$, respectively. Then we have

$$k_\lambda^*(\xi)=\left(1-\frac{|\xi|}{2\lambda}\right)k^*(\xi), \qquad |\xi|\leqq 2\lambda;$$
$$=0, \qquad |\xi|>2\lambda.$$

By the Plancherel's theorem

$$\int_{-\infty}^{\infty}|K^*(\eta)-K_\lambda^*(\eta)|^2d\eta=\int_{-\infty}^{\infty}|k^*(\xi)-k_\lambda^*(\xi)|^2d\xi$$
$$=\int_{-\infty}^{-2\lambda}+\int_{2\lambda}^{\infty}|k^*(\xi)|^2d\xi+\frac{1}{(2\lambda)^2}\int_{-2\lambda}^{2\lambda}|\xi k^*(\xi)|^2d\xi,$$

which tends to zero as $\lambda\to\infty$. Thus the lemma is proved.

3.[2] We will now prove Theorem 3. Instead of (1), it is sufficient to prove that

$$\lim_{n\to\infty}\int_0^{\infty}f\left(x+\frac{\xi}{n}\right)K(\xi)d\xi=f(x)\int_0^{\infty}K(\xi)d\xi. \tag{4}$$

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1) We can prove this lemma in more elementary manner as Lemma 67 in Wiener's work: Fourier Integral and Certain of its Applications.

2) Cf. S. Bochner: Berliner Sitzber., 1933.

And we may suppose that $f(x)=0$. Further, by Theorem 1, we may suppose that $f(\xi)$ is identically zero in the neighbourhood of x. If we put $\xi=e^{\eta}$, $n=e^{\nu}$ and

$$f\left(x+\frac{\xi}{n}\right)=h(\eta-\nu),$$

then (4) becomes

$$\lim_{\nu\to\infty}\int_{-\infty}^{\infty}h(\eta-\nu)K(e^{\eta})e^{\eta}d\eta=0. \tag{5}$$

If we put $K^*(\eta)=K(e^{\eta})e^{\eta}$, then (5) becomes

$$\lim_{\nu\to\infty}\int_{-\infty}^{\infty}h(\eta-\nu)K^*(\eta)d\eta=0. \tag{6}$$

4. We have

$$\int_{-\infty}^{\infty}h(\eta-\nu)K_\lambda^*(\eta)d\eta=\frac{1}{\pi}\int_{-\infty}^{\infty}h(\eta-\nu)d\eta\int_{-\infty}^{\infty}K^*(\xi)\frac{\sin^2\lambda(\xi-\eta)}{\lambda(\xi-\eta)^2}d\xi$$
$$=\frac{1}{\pi}\int_{-\infty}^{\infty}K^*(\xi)d\xi\int_{-\infty}^{\infty}h(\eta-\nu)\frac{\sin^2\lambda(\xi-\eta)}{\lambda(\xi-\eta)^2}d\eta,$$

the inversion of the order of integrals being permissible by the absolute convergence of the repeated integral. By Lemma 1,

$$\lim_{\nu\to\infty}\int_{-\infty}^{\infty}h(\eta-\nu)K_\lambda^*(\eta)d\eta=0.$$

On the other hand,

$$\left|\int_{-\infty}^{\infty}h(\eta-\nu)K_\lambda^*(\eta)d\eta-\int_{-\infty}^{\infty}h(\eta-\nu)K^*(\eta)d\eta\right|$$
$$=\left|\int_{-\infty}^{\infty}h(\eta-\nu)[K_\lambda^*(\eta)-K^*(\eta)]d\eta\right|$$
$$\leq\left\{\int_{-\infty}^{\infty}|h(\eta-\nu)|^2d\eta\int_{-\infty}^{\infty}|K_\lambda^*(\eta)-K^*(\eta)|^2d\eta\right\}^{\frac{1}{2}}$$
$$=\left\{\int_{-\infty}^{\infty}|h(\eta)|^2d\eta\int_{-\infty}^{\infty}|K_\lambda^*(\eta)-K^*(\eta)|^2d\eta\right\}^{\frac{1}{2}},$$

which tends to zero as $\lambda\to\infty$, by Lemma 2. Thus (6) and then the theorem is proved.