66. A Remark on an Integral Equation.

By Shin-ichi IZUMI.

Mathematical Institute, Tohoku Imperial University, Sendai.

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Mr. Nagumo has proposed the problem to solve the integral equation

$$f(x)=\frac{1}{2}\int_{x-1}^{x+1}f(t)dt. \qquad (1)$$

In this paper two types of solutions are found. The first is of exponential type and the second is that belonging to L^2-class.

Theorem 1. If $f(x)$ is a solution of (1) such that

$$f(x)=O(e^{A|x|}),$$

A being a positive number, $f(x)$ is of the form

$$A'x+B+\sum ae^{-u^*x},$$

where u^* is the non-zero root of the equation

$$1=\frac{e^u-e^{-u}}{2u}, \qquad (2)$$

such that $|R(u^*)|<A$ and A', B, a are arbitrary constants.

Proof. If we put

$$K(x)=\frac{1}{2}, \quad |x|\leqq 1; \qquad K(x)=0, \quad |x|>1,$$

then (1) becomes

$$f(x)=\int_{-\infty}^{\infty}K(x-t)f(t)dt.$$

Therefore we can apply the theory of Hopf and Wiener.[1] As easily be seen, (2) has the origin as only one double zero. Thus we get the theorem.

Theorem 2.[2] If $f(x)$ is a solution of (1) belonging to L^2-class in $(-\infty, \infty)$, then $f(x)$ is identically zero.

Proof. We have

$$\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}f(x)e^{-uxi}dx=\frac{1}{2\sqrt{2\pi}}\int_{-A}^{A}e^{-uxi}dx\int_{x-1}^{x+1}f(t)dt$$

$$=\frac{1}{2\sqrt{2\pi}}\Big[\int_{-(A-1)}^{A-1}f(t)dt\int_{t-1}^{t+1}e^{-uxi}dx$$

$$+\int_{-A-1}^{-A+1}f(t)dt\int_{-A}^{t-1}e^{-uxi}dx+\int_{A-1}^{A+1}f(t)dt\int_{t-1}^{A}e^{-uxi}dx\Big]$$

1) E. Hopf and N. Wiener: Sitzungsberichte der Preussischen Akademie, 1931. Cf. E. Hopf: ibid., 1928, and Paley-Wiener: Fourier transforms in the complex domain, 1934, Chapter IV.

2) Cf. Hardy-Titchmarsh: Proc. London Math. Soc., (2) **23** (1924) and **30** (1930).

$$= \frac{-1}{2\sqrt{2\pi}\,ui}\Big[\int_{-(A-1)}^{A-1} f(t)e^{-uti}(e^{-ui}-e^{ui})dt$$

$$+\int_{-A-1}^{-A+1} f(t)\{e^{-u(t-1)i}-e^{uAi}\}dt-\int_{A-1}^{A+1} f(t)\{e^{-Aui}-e^{-u(t-1)i}\}dt\Big].$$

$$\int_{-\infty}^{\infty}\Big|\frac{1}{u}\int_{-A-1}^{-A+1} f(t)\{e^{-u(t-1)i}-e^{uAi}\}dt\Big|^2 du$$

$$\leqq K\int_{-\infty}^{\infty}\frac{du}{1+u^2}\Big[\int_{-A-1}^{-A+1}|f(t)|\,dt\Big]^2 \leqq 2K\int_{-A-1}^{-A+1}|f(t)|^2 dt\int_{-\infty}^{\infty}\frac{du}{1+u^2},$$

K being an absolute constant. Therefore we have

$$\underset{A\to\infty}{\mathrm{l.i.m.}}\ \frac{1}{u}\int_{-A-1}^{-A+1} f(t)\{e^{-u(t-1)i}-e^{uAi}\}dt=0.$$

Similarly

$$\underset{A\to\infty}{\mathrm{l.i.m.}}\ \frac{1}{u}\int_{A-1}^{A+1} f(t)\{e^{-Aui}-e^{-u(t-1)i}\}dt=0.$$

If $F(u)$ is the Fourier transform of $f(x)$, then we have

$$F(u)=\frac{e^{-ui}-e^{ui}}{-2ui}F(u)$$

for almost all u. Hence $F(x)$, and then $f(x)$, is equivalent to zero. The solution of (1) is continuous, therefore $f(x)$ is identically equal to zero. Thus the theorem is proved.

In Theorem 2, we can replace L^2-class by L^p-class $(1<p\leqq 2)$. In this case it is sufficient to use the Titchmarsh's theorem instead of the Plancherel's theorem.