THE SERIES OF SETS OF POINTS ON AN ALGEBRAIC SURFACE.

By

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1. In several papers published in the last four years[1)] I have given the theory of the series of sets of points on an algebraic surface. The creation of such a theory was necessary in order to know most deeply the substantial properties of the algebraic surfaces, invariant under birational transformations.

The most important series of sets of points upon algebraic surfaces are the (so called by me) *series of equivalence.* Let us first define an *elementary series* or a *series of intersection* as a series of sets of niveau-points for two rational functions of a variable point on the given surface F, namely as a series of sets of points of intersection of two curves belonging to two linear systems $\sum_1$, $\sum_2$, outside perhaps of some fixed intersections. When the dimensions of $\sum_1$, $\sum_2$ are null, we have one set of points; when only one of the two systems consists

1) See the following papers: Severi, La serie canonica e la teoria delle serie principali di gruppi di punti sopra una superficie algebrica. (Commentarii Mathematici Helvetici, 1932); Un nuovo campo di ricerche nella geometria sopra una superficie e sopra una varietà algebrica. (Memorie della R. Accademia d'Italia, 1932); Quelques théories nouvelles en géométrie algébrique. (C. R., Paris, 1933); Nuovi contributi alla teoria delle serie di equivalenza sulle superficie e dei sistemi di equivalenza sulle varietà algebriche. (Mem. Accad. Ital., 1933); La teoria delle serie di equivalenza e delle corrispondenze a valenza sopra una superficie algebrica, 7 note. (Rendiconti Accad. naz. Lincei, 1933); La base per le varietà algebriche di dimensione qualunque contenute in una data e la teoria generale delle corrispondenze fra i punti di due superficie algebriche. (Mem. Accad. Ital., 1934); La théorie générale des correspondences entre deux surfaces algébriques. (C. R., Paris, 1934); Le involuzioni razionali sopra una superficie come serie di equivalenza, 2 Note. (Rendiconti Accad. naz. Lincei, 1934); Caratterizzazione geometrica, topologica e trascendente delle serie di equivalenza sopra una superficie. (Rendiconti Accad. naz. Lincei, 1934); Ancora sulla caratterizzazione topologica e trascendente delle serie di equivalenza. (Rendiconti Accad. naz. Lincei, 1934); Un' altra proprietà fondamentale delle serie di equivalenza sopra una superficie. (Rendiconti Accad. naz. Lincei, 1935); Equivalenza funzionale d'una curva come gruppo virtuale parziale d'una serie d'equivalenza sopra una superficie. (Volume di scritti offerti a L. Berzolari, Pavia, 1936). The part of the new theory that I have developed on the first five quoted papers is resumed in Zariski's book, Algebraic surfaces (Berlin, Springer, 1935).

of one curve C, we have on C a *curvilinear series of equivalence*, whose sets are each a sum of sets belonging to some linear series on the irreducible components of C; when the dimensions of $\sum_1$, $\sum_2$ are both greater than zero, we have a *superficial elementary series.*

It may be that such a set of points belonging to the same component of two curves of $\sum_1$, $\sum_2$ exists, and that it is not possible to define it by intersection of these curves. It may be, however, that it is possible to obtain this set as partial or total limit of a suitable set of intersection of two variable curves of $\sum_1$, $\sum_2$. We will include every limit set of any set of intersection of two variable curves of $\sum_1$, $\sum_2$, in the elementary series, which becomes also *algebraic* and even a *rational* series.

A *series of equivalence* of F is a continuous system of sets obtained by addition and subtraction from the sets belonging to a finite number of elementary series. Some points of a variable set on a series of equivalence may be *fixed points*; others may be variable points on a curve (irreducible or reducible) of F, forming together a variable set on a series of equivalence on the curve. We call such a set a *semifixed set.*

Concerning the definition of the series of equivalence on F, we observe:

1) The definition becomes most useful when one extends the field of the sets of points by introducing the *virtual finite sets*, according to a conception which I have introduced in algebraic geometry many years ago. A symbol as $-A$ or $A-B$, where A, B are effective finite sets of points, is called a virtual finite set: it represents a potential subtraction or a potential addition followed by a subtraction applied to an effective suitable set of points. This is an analogous conception to the negative numbers in an arithmetic field.

2) It may be that an algebraic curve on F must be *virtually* considered as a set of points of a given series σ of equivalence or as a set partially contained in σ. That occurs when the curve holds a series of sets such that each of these sets may be obtained as limit of a set partially or totally contained in the general set of σ. The limit sets form a series of equivalence on the curve, and each of those gives the so-called *functional equivalence* of the curve as a partial or total set of σ.

2. A (algebraic, irreducible) series of sets of points of F has generally three dimensions: the *superficial dimension* $2r$, where r (≥ 0) is the highest number of points that may be chosen arbitrarily on F

for a set of the series; the *curvilinear dimension* s, where $s\ (\geq 0)$ is the highest number of points that may be chosen arbitrarily for a set of the series on the curve holding the sets which pass through r arbitrary points of F; the *total dimension* $2r+s$, giving the number of parameters of a variable set on the series. When $s=0$, the series is *totally superficial*; when $r=0$, it is a curvilinear series.

3. For better knowledge of new properties that I shall expound later, I will first give a resumé of some properties contained in my preceding papers.

a) *Geometrical characteristic properties for a series of equivalence.*

I. A necessary and sufficient condition that a given series of sets of points upon a surface F, belonging to a linear space S_r, be a series of equivalence, is that it may be cut out on F by a continuous system of algebraic $(r-2)$-dimensional varieties, outside of eventual fixed points and semifixed sets.

II. A necessary and sufficient condition that a given series on F be a series of equivalence, is that it is a *rational* series of virtual sets of points.

b) *Topological characteristic properties for a series of equivalence.*

Let σ be an algebraic series of sets of points on F and let V be a variety whose points represent birationally the elements (sets) of the series. We say that σ is a *series of linear circulation zero* when the cycle on F corresponding to every linear cycle on V is a null linear cycle or a divisor of zero.[1]

We say that σ is a *series of algebraic circulation* when the cycle on F corresponding to every superficial cycle on V is an algebraic cycle.

Finally we say that σ is a *series of null cyclo-torsion* when the cycle on F corresponding to every superficial divisor of zero on V is a null cycle.

I have proved that:

I. A series of equivalence is a series of linear circulation zero, of algebraic circulation and of null cyclo-torsion.

I have not been able till now to answer the question of whether the reciprocal proposition is true or not. But it is possible to invert the greatest part of the theorem in the following manner.

Let us observe first that the multiple by a fixed integer >1 of the sets of a series, may form a series of equivalence and the series

1) These series are first considered by Albanese (see Zariski's book).

may not be a series of equivalence. We call such a series a *series of pseudoequivalence.*

In the theory of linear series on an algebraic curve one does not find an analogous property; namely every pseudolinear series on a curve is a linear series.

The real reason is that the Riemannian variety of an algebraic variety of dimension >1 may possess a topological torsion, while every Riemann surface is deprived of torsion.

II. A necessary and sufficient condition that a series of sets of points on F be a series of pseudoequivalence, is that it be a series of linear circulation zero and of algebraic circulation.

A series of pseudoequivalence deprived of torsion on a surface without torsion is a series of equivalence.

III. A necessary and sufficient condition that the series of the points of a surface be a series of pseudoequivalence, is that the surface is regular (namely with arithmetic genus = geometric genus) and that its geometric genus is zero.

For characterizing a rational surface among such surfaces it is necessary also to adjoin other properties, that I have given in another paper.

c) *Transcendental characteristic properties for a series of equivalence.*

The linear series on a curve are characterized by the transcendental point of view by the Abel's theorem. The research of analogous properties for the series of equivalence presents many new serious difficulties. I have given the following theorems.

I. A necessary and sufficient condition that a series on a surface F be a series of pseudoequivalence is that every linear or quadratic differential form of the first kind attached to F give a sum identically zero on the sets of the series.

When only the sums relative to the linear differential forms of the first kind are null, the series is of linear circulation zero[1]; while if only the sums relative to the quadratic differential forms of the first kind are zero, the series is of algebraic circulation.

Let F be a surface in ordinary space S_3 and let x, y, z be Cartesian coordinates. A necessary and sufficient condition that a set (x_1, y_1, z_1), ..., (x_n, y_n, z_n) of n variable points on F, describes a series of linear circulation zero is

1) The property was first given by Albanese.

$$(1) \quad \sum_{h=1}^{n} A_k(x_h, y_h, z_h)dx_h + B_k(x_h, y_h, z_h)dy_h = 0 \qquad (k=1, 2, \ldots\ldots, q),$$

where $A_k dx + B_k dy$ is a total differential of the first kind attached to F and q is the irregularity of the surface.

A necessary and sufficient condition that the given set describes a series of algebraic circulation is

$$(2) \qquad \sum_{h=1}^{n} \varphi_l(x_h, y_h, z_h)dx_h dy_h = 0 \qquad (l=1, 2, \ldots\ldots, p),$$

where $\iint \varphi_l dx dy$ is a double integral of the first kind attached to F and p is the geometric genus.

If the set satisfies both systems (1), (2) it describes a series of pseudoequivalence.

4. The theory of the differential form of every order, considered first by Poincaré, have been developed by E. Cartan and others. E. Kähler[1] published recently on this argument a nice and ingenious little book. By applying this theory to (1), (2), I have now obtained some important results concerning the dimension of the most ample (complete) series of linear circulation zero or of algebraic circulation or finally of pseudoequivalence. I propose to expose here those results, by reserving the detailed proofs to another ampler paper.

I. Let us remember that the so called "Severi's series of equivalence" is the series formed by the Jacobian sets of the simple integrals of the first kind attached to F.[2] I have given also, besides this definition, two geometrical definitions of this series.[3] We will say that i is the *first index of speciality* of a given set G_0 of n points of F, when $i-1$ is the dimension of the totality of the sets of the Severi's series passing through G_0.

In order to determine the complete series of linear circulation zero holding G_0, it is necessary to consider first the *algebraic* variety W of all sets of n points of F having the first index i, and to study afterwards the differential system (1), where the sets $(x_1, y_1, z_1), \ldots\ldots,$

1) Einführung in die Theorie der Systeme von Differentialgleichungen (Hamburger mathematische Einzelschriften, 16 Heft, Leipzig, 1934).

2) See for ex. Zariski's book, p. 179.

3) See the first and the second of my quoted papers. The first definition is essentially in relation to the irregularity of the surface; the second may be applied to every surface. *A priori* the first definition gives on an irregular surface an uncomplete series but from the following theorems one deduces that the two definitions lead, on an irregular surface, to the same complete series.

(x_n, y_n, z_n) varies on W in the neighborhood of the set $G_0\{(x_1^0, y_1^0, z_1^0), \dots\dots, (x_n^0, y_n^0, z_n^0)\}$.

We may represent, without restrictions, the neighborhood of G_0 on the totality of all sets of n points of F, with the neighborhood of the point $P_0(x_1^0, y_1^0, \dots\dots, x_n^0, y_n^0)$ on $2n$-dimensional linear space $(x_1, y_1, \dots\dots, x_n, y_n)$.

Then the neighborhood of G_0 on W is represented in this space by an analytical branch having the origin P_0.

The 1-dimensional integral elements[1] of the system (1), issued by P_0 are $\infty^{2n-1-(q-i)}$. I have proved that a generic of such elements is a *regular integral element*[2] and therefore by applying in a recurrent manner the proposition given on p. 32 of Kähler's book, I have arrived at the following theorem:

Let G_0 be a set of n points on a surface F and let i be the first index of speciality of G_0. Then this set is contained only on one complete algebraic series of linear circulation zero and of order n, whose generic set have the same index of speciality and whose total dimension ρ is equal to $2n-q+i$, where q is the irregularity of the surface.[3]

The enunciation is not exactly analogous to the enunciation of Riemann-Roch's theorem on algebraic curves, because it is not certain that all sets of the most ample series, whose generic set have the first index of speciality i, have the same index. Indeed we are not able to exclude that any particular sets of the series may have an index greater than i.

By considering the system (1) upon the variety W, a set G_0' of W whose first index i' is greater than i, is not a regular 0-dimensional integral element of (1); but it is a regular 0-dimensional integral element of (1) upon the variety W' whose generic set have the first index i'. Therefore G_0' determine on W' one series of linear circulation zero, whose total dimension is $\rho'=2n-q+i'$ and which intersects the preceding ρ-dimensional series into a series holding G_0'.

II. For the complete determination of an integral variety of the differential system (2), which characterizes the series of algebraic circulation, *one* set of n points is not sufficient, because substantially the system (2) is equivalent to a system of partial differential equations and therefore it must give something which is equivalent to some arbitrary functions.

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1) See Kähler's book, p. 15.

2) Kähler, l. c., p. 23.

3) The inequality $\rho \geq 2n-q$ was first enunciated, without proof, by Albanese (Bollettino Unione matem. ital., 1933).

I will define first the *second index of speciality* j of a set G_0 of n points of F. The second index j is the number of linearly independent canonic curves of the surface, which pass through G_0. We must consider the system (2), where the variable set belongs to the *algebraic* variety Z of all sets of n points of F whose generic set have the second index j. By representing the neighborhood of G_0 on Z with a certain analytical branch of the space $(x_1, y_1, \cdots\cdots, x_n, y_n)$, having its origin in P_0, I have found that the 2-dimensional integral elements of the system (2), issued by a generic direction with the origin P_0, are $\infty^{2(n-2)-(p-j)}$ and that a generic of such elements is a *regular integral element* of (2). By applying the same proposition above quoted I have obtained the following theorem:

Let s denote a one-dimensional algebraic series of sets of n points on a surface F and let j be the second index of speciality of a generic set G_0 of the series. Then the series s determine an algebraic complete series of algebraic circulation of n points, which holds s and whose generic set have the same second index j. The complete series have the total dimension $2n-p+j$, where p is the geometric genus of F.

The algebricity of the most ample integral varieties of each of the systems (1), (2) is a consequence of the following proposition: "An analytical variety, belonging to any space, whose singular points are all algebroid points, is an algebraic variety."

Indeed I prove that each of the systems (1), (2), in the neighborhood of G_0, is equivalent to a system of a finite number of equations, whose first members are analytical *regular* functions of the variable set G.

Relatively to the particular sets of the complete series we are not able to exclude that they have a second index of speciality greater than j.

Finally the combination of the two last theorems give the following:

III. *Let s denote a one-dimensional algebraic series of order n and of linear circulation zero on a surface F, and let i, j be the first and the second index of speciality of the generic set of s. Then the series s is contained in a complete algebraic series of pseudoequivalence of the same order n, whose generic set have the same index i, j and whose total dimension is $2n-p-q+i+j$.*

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