12. *On the Univalency and Multivalency of a Class of Meromorphic Functions.*

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1. *Theorem.*

Definition. Let z be a complex variable. We say that a domain is a *fan-shaped*, if it is given by the following expression:

$\theta_1 \leq \arg z \leq \theta_2$ (θ_1, θ_2 are two arbitrary angles such as $\theta_1 \leq \theta_2$);

$r_1 \leq |z| \leq r_2$ (r_1, r_2 are two arbitrary real numbers such as $0 \leq r_1 \leq r_2$).

We consider also as special case the figure obtained by putting $r_2 = \infty$ in the above expression.

Theorem. *Consider a function* $f(z) = \frac{a}{z} + g(z)$ *defined in a certain convex domain* A, *where* $g(z)$ *is regular in the domain* A *and* a *is an arbitrary constant. Let* p *be a positive integer. Suppose that*

(1°) $g^{(p)}(z)$ $(z \in A)$ *is contained in a convex domain* $\mathfrak{A}$,

(2°) *there exist a fan-shaped domain* B *such that the image* $\mathfrak{B}$ *of* B *transformed by the function* $w = \frac{(-1)^{p+1} p! a}{z^{p+1}}$ *is disjoint from* $\mathfrak{A}$: $\mathfrak{A} \cdot \mathfrak{B} = 0$. *Then* $f(z)$ *is at most* p*-valent in the common part of* A *and* B: $A \cdot B$.

Remark. Evidently, the domain $\mathfrak{B}$ is also fan-shaped and can be easily constructed from B.

Lemma. P. Montel[1] has proved the following lemma:

Be $g(z)$ *a function which is regular in a certain convex domain* A. *Let* $z_1, z_2, \dots, z_p, z_{p+1}$ *be* $p+1$ *arbitrary points of* A. *Consider the following expressions*

$$\Delta_0(z_1) = g(z_1), \qquad \Delta_1(z_2, z_1) = \frac{g(z_2) - g(z_1)}{z_2 - z_1}, \qquad \dots\dots\dots\dots,$$

$$\Delta_p(z_{p+1}, z_p, \dots, z_1) = \frac{\Delta_{p-1}(z_{p+1}, z_{p-1}, \dots, z_1) - \Delta_{p-1}(z_p, z_{p-1}, \dots, z_1)}{z_{p+1} - z_p}.$$

Then $p! \Delta_p(z_{p+1}, z_p, \dots, z_1) \in \mathfrak{A}$ *where* $\mathfrak{A}$ *is a convex domain which contain all the points* $g^{(p)}(z)$, $z \in A$.

Proof of the Theorem. We take $p+1$ arbirary points $z_1, z_2, \dots, z_{p+1}$ in $A \cdot B$ and we consider the following expressions $\bar{\Delta}_0, \bar{\Delta}_1, \dots, \bar{\Delta}_p$:

1) P. Montel: Annali R. Scuola normale super. di Pisa, 2 serie, 1, 1932, p. 371–384; and Comptes Rendus, t. 201, 1935, p. 322–324.

$$\bar{\Delta}_0(z_1)=f(z_1)\,,\qquad \bar{\Delta}_1(z_2, z_1)=\frac{f(z_2)-f(z_1)}{z_2-z_1}\,,\qquad \cdots\cdots\cdots\cdots,$$

$$\bar{\Delta}_p(z_{p+1}, z_p, \dots, z_1)=\frac{\bar{\Delta}_{p-1}(z_{p+1}, z_{p-1}, \dots, z_1)-\bar{\Delta}_{p-1}(z_p, z_{p-1}, \dots, z_1)}{z_{p+1}-z_p}\,.$$

Then we have the identity:

$$p\,!\,\bar{\Delta}_p(z_{p+1}, z_p, \dots, z_1)=-\left[\frac{(-1)^{p+1}p\,!\,a}{z_{p+1}, z_p, \dots, z_1}-p\,!\,\Delta_p(z_{p+1}, z_p, \dots, z_1)\right].$$

On the other hand, we can easily see from the assumption (2°) that

$$\frac{(-1)^{p+1}p\,!\,a}{z_{p+1}, z_p, \dots, z_1}\in\mathfrak{B}$$

and by our Lemma that

$$p\,!\,\Delta_p(z_{p+1}, z_p, \dots, z_1)\in\mathfrak{A}\,.$$

Then the assumption $\mathfrak{A}\cdot\mathfrak{B}=0$ gives the result

$$\frac{(-1)^{p+1}p\,!\,a}{z_{p+1}, z_p, \dots, z_1}\neq p\,!\,\Delta_p(z_{p+1}, z_p, \dots, z_1)\qquad\text{viz.}\qquad \bar{\Delta}_p(z_{p+1}, z_p, \dots, z_1)\neq 0\,.$$

Thus $f(z)=\frac{a}{z}+g(z)$ is at most p-valent in $A\cdot B$.

2. *Special cases.*

(1) Consider the case where the domain B is given by $|z|\leq\rho$, then $\mathfrak{B}$ is given by $|w|\geq\frac{p\,!}{\rho^{p+1}}|a|$. We obtain from our Theorem

Corollary 1.[1] *Consider the function* $f(z)=\frac{1}{z}+g(z)$ *where* $g(z)$ *is regular in a convex domain* A. *If* $|g^{(p)}(z)|<\frac{p\,!}{\rho^{p+1}}$ *for* $z\in A$, *then* $f(z)$ *is at most p-valent in the common domain of* $|z|\leq\rho$ *and* A.

(2) Consider the case where the domain B is given by $|z|\geq\rho$, then $\mathfrak{B}$ is given by $|w|\leq\frac{p\,!}{\rho^{p+1}}|a|$. We obtain from our Theorem

Corollary 2. *Let* $g(z)$ *be a function which is regular in a certain convex domain* A. *If* $R[e^{i\theta}g^{(p)}(z)]>\frac{p\,!}{\rho^{p+1}}$ *for a fixed real number* θ *and* $z\in A$, *then* $f(z)=\frac{1}{z}+g(z)$ *is at most p-valent in the common part of* $|z|\geq\rho$ *and* A.

For $p=1$, we have thus a new proof of the theorem du to T. Sato.[2]

(3) Let us put $a=0$ and take the whole z-plane as domain B. Then $\mathfrak{B}$ is reduced to a single point viz. the origin of w-plane. There-

1) See K. Kimura: Osaka Shijodanwakai No. 30, p. 1–6.
2) See Sato: Proc. **11** (1935), 212–213.

fore we can take as $\mathfrak{A}$ the half-plane limited by a straight line which passes through the origin of w-plane. Thus we have the following

Corollary 3. *Suppose that $f(z)$ is regular in a certain convex domain A. If $R[e^{i\theta}f^{(p)}(z)]>0$ for a fixed real number θ and $z\in A$, then $f(z)$ is at most p-valent in the domain A.*

namely a theorem of Osaki; the case for $p=1$ was given by Wolff and Noshiro.[1)]

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1) See Osaki: Science Report of the Tokyo Bunrika Daigaku, 2, A, 1935, p. 167–188. Wolff: Comptes Rendus, t. 198, 1934, p. 1209. Noshiro: Journal of the Faculty of Science, the Hokkaido Imperial University, 2, 1934, p. 129–155.