PAPERS COMMUNICATED

9. *Some Theorems on a Cluster-set of an Analytic Function.*

By Kiyoshi NOSHIRO.
Mathematical Institute, Hokkaido Imperial University, Sapporo.

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1. Let $f(z)$ be uniform and meromorphic in a finite connected domain D. We shall first state some notations—$\mathfrak{D}$: the value-set of $f(z)$ in D, F: the boundary set of $\mathfrak{D}$, H: the set of all cluster values[1] at the boundary of D, $\overline{M}$: the closure of M, CM: the complementary set of M. It is evident that $F \subset H \subset \overline{\mathfrak{D}}$ and they are all closed sets. In general the equality $F=H$ does not hold. For example, if we take $w=f(z)=z^2$ and $D: 0<\arg z<\frac{3\pi}{2}$, $R_1<|z|<R_2$, then $\mathfrak{D}$ is a ring: $R_1^2<|w|<R_2^2$ and H consists of two segments $(-R_2^2, -R_1^2)$, (R_1^2, R_2^2) and two circles $|w|=R_1^2$, $|w|=R_2^2$. Now suppose that $F=H$. Then we see easily that for any value $\alpha \in \mathfrak{D}$, $f(z)$ never takes α at infinite times, for otherwise α would be a cluster value, so that α would belong to $F=H$. This is a contradiction. Next we shall show that $f(z)$ is *exactly p-valent* in D, if a certain value $\alpha \in \mathfrak{D}$ is taken p times. Consider a closed circular domain $\overline{K}$ contained entirely interior to $\mathfrak{D}$. The set of points z, each of which has an image in $\overline{K}$, in general, consists of a finite or an enumerable infinity of connected domains $\overline{\Delta}_i$ in D. However, since $H=F$, each $\overline{\Delta}_i$ must lie completely in the interior of D and so the number of $\overline{\Delta}_i$ is finite. Then $f(z)$ takes in D any value $\alpha \in K$ exactly at the same number of times, say p times, since this holds in each Δ_i by the principle of arguments. Now, let α and β be two finite points in $\mathfrak{D}$. Then we can find a finite sequence of closed circular discs, $\overline{K}_0, \overline{K}_1, \ldots\ldots, \overline{K}_n$ such that each $\overline{K}_i \subset \mathfrak{D}$, $\alpha \in K_0$, $\beta \in K_n$ and $K_i \cdot K_{i+1} \neq 0$ where $i=0, 1, \ldots\ldots, n-1$. Hence $f(z)$ takes α and β at the same number of times, then $f(z)$ is *exactly p-valent* in D, i. e. $f(z)$ *takes in D any value p times.* Conversely, if $f(z)$ is exactly p-valent, then it follows that $H=F$. Let α be an arbitrary finite value in $\mathfrak{D}$ and a_i be an α-point of order p_i. If there are n α-points in total, then clearly $p=\sum_{i=1}^{n} p_i$. Let $\overline{K}_i$ be a small circle: $|z-a_i| \leqq \rho$, lying within D, such that $\overline{K}_j \cdot \overline{K}_{j'}=0$ $(i \neq j')$, and denote by $\mathfrak{D}_i$ the value-set of $f(z)$ in K_i. Then there is a circle $C: |w-\alpha|<\sigma$, contained in $\prod_{i=1}^{n} \mathfrak{D}_i$, any value of which can be taken at least p_i times in each K_i $(i=1, 2, \ldots\ldots, n)$, provided that σ is sufficiently small. Consequently it follows that α cannot be a cluster-value, for otherwise there be a point $z' \in D-\sum_{i=1}^{n} \overline{K}_i$ such

1) We call α a cluster value of $f(z)$ at $z=\zeta$, if there exists a sequence $z_n \to \zeta$, $z_n \neq \zeta$, $z_n \in D$, such that $f(z_n) \to \alpha$.

that $f(z')$ lies in C, then $w'=f(z')$ could be taken at least $p+1=\sum_{i=1}^{n} p_i+1$ times in D and this is contrary to the assumption. If the infinity belongs to $\mathfrak{D}$, we see that it cannot also be a cluster value, repeating a similar argument to the above. Thus we have

Theorem 1. *Let $f(z)$ be uniform and meromorphic in a finite domain D. Suppose $H=F$. Then $f(z)$ is exactly p-valent in D, if a certain value is taken p times. Conversely if $f(z)$ is exactly p-valent in D, then it follows that $H=F$.*

Let D be *simply connected.* Make a sequence of polygonal domains D_n, such that $\overline{D}_n \subset D_{n+1}$, $D_n \to D$ and each D_n is simply connected, and denote the value-set of $f(z)$ for $D-D_n$ by Δ_n. Then the closure $\bar{\Delta}_n$ is a continuum. Therefore H is a continuum, since $H=\prod_{n=1}^{\infty} \bar{\Delta}_n$. Consequently, if $H=F$, the value-set $\mathfrak{D}$ is simply connected. Thus we have a precise form of Mr. Satô's theorem:[1]

Theorem 2. *Let $f(z)$ be regular in a finite simply connected domain D and let $f'(z) \neq 0$ in D. Then $f(z)$ is univalent in D, provided that $H=F$.*

Any inverse element $P(w-a)$, $a \in \mathfrak{D}$ is analytically continuable along any way contained in $\mathfrak{D}$, where $\mathfrak{D}$ is simply connected as we remarked above. Applying a known theorem "Monodromiesatz," the analytic function determined by the element $P(w-a)$ is uniform in $\mathfrak{D}$, and so $f(z)$ is univalent in D. By a similar argument, we obtain

Theorem 3. *Let $f(z)$ be a uniform regular function in a finite domain D, such that $f'(z)$ never vanishes in D, and suppose that the value-set $\mathfrak{D}$ is simply connected. Then D is simply connected, provided that $H=F$.*

In other words, we get

Theorem 4. *Let $f(z)$ be uniform and regular in a finite domain D. Suppose that $H=F$ and that the value-set $\mathfrak{D}$ is simply connected. Then $f'(z)$ has at least one zero-point in D, provided that D is multiply connected.*

For example; consider $w(z)=z+\dfrac{1}{z-a}+\dfrac{1}{z-b}$, $a \neq b$. Then the level curve: $|w(z)|=R$ consists of three simple closed regular curves if R is sufficiently large. Consequently the finite domain D enclosed by these curves is triply connected and the value-set $\mathfrak{D}$ is a circular domain $|w|<R$. In this case $w(z)$ is exactly 3-valent in D and $w'(z)$ vanishes there.

2. In the previous case, H does not contain any interior point, since $H=F$ and F is the boundary of $\mathfrak{D}$. But, if $H \neq F$, then H may contain an interior point. Here we will obtain a necessary condition that H should have an interior point. Let a_0 be an interior point of H and K be a circular domain with centre a_0 lying entirely within H. It is clear that $K\cdot\mathfrak{D} \neq 0$, since every $a \in K$ is a cluster value. Select a point a_1 within $K\cdot\mathfrak{D}$, such that $a_1=f(z_1)$, $f'(z_1) \neq 0$, then the inverse regular

1) This journal, vol. 12, p. 332.

element $z=P(w-a_1)$ maps the circle $\bar{K}_1$: $|w-a_1| \leqq \rho_1$, contained within K, conformally on a closed simply connected domain $\bar{\Delta}_1$, if ρ_1 is sufficiently small. Next, since every $a \in K_1$, which denotes the interior of $\bar{K}_1$, is a cluster value, we can select a point a_2 within K_1, such that $a_2=f(z_2)$, $f'(z_2) \neq 0$ and $z_2 \in D-\bar{\Delta}_1$, then the element $z=P(w-a_2)$ maps $\bar{K}_2$: $|w-a_2| \leqq \rho_2$, contained within K_1, conformally on a closed domain $\bar{\Delta}_2$ interior to $D-\bar{\Delta}_1$, if ρ_2 is sufficiently small. Repeating this process, we have two sequences $\{\bar{K}_n\}$ and $\{\bar{\Delta}_n\}$ with the following properties: $\bar{K}_n \subset K_{n-1}$ and $\bar{\Delta}_n \subset D-(\bar{\Delta}_1+\bar{\Delta}_2+\cdots\cdots\bar{\Delta}_{n-1})$ and $\bar{K}_n$ is the image of $\bar{\Delta}_n$ by $w=f(z)$. Consequently $f(z)$ takes every value, belonging to $\prod_{n=1}^{\infty} \bar{K}_n$, at infinite times. Thus we have

Theorem 5. *Let $f(z)$ be uniform and meromorphic in D and let H be the cluster-set. Then $f(z)$ takes some values at infinite times, provided that H contains an interior point.*

More precisely, if we denote by Γ the set of values taken at infinite times, then $I(H) \subset \bar{\Gamma}$, where $I(H)$ means the interior part of H.

As an immediate result, we get

Theorem 6. *Under the same condition as in theorem 5, if $f(z)$ does not take any value infinite times, then the set H contains no interior point.*

Next, suppose that H does not contain any interior point. Since $\Delta=\bar{\mathfrak{D}}-H=(\mathfrak{D}+F)\,CH=\mathfrak{D}\cdot CH+F\cdot CH=\mathfrak{D}\cdot CH$, Δ is an open set and can be decomposed into its components Δ_i $(i=1, 2, \ldots\ldots)$, each of which is a connected domain. By the same argument in the proof of theorem 1, any value of each Δ_i is taken by $f(z)$ at the same (finite) number of times, say p_i times. Denote the maximum of p_i by p and let p be finite. Then $f(z)$ takes every value at most p times in D. Evidently we have only to consider a value belonging to $\mathfrak{D}\cdot H$. Let a be such a value. If $f(z)$ takes a at least $p+1$ times in D, then, describing a small circle K with centre a, every value, lying within K, can be taken by $f(z)$ at least $p+1$ times in D, while $K\cdot\sum \Delta_i \neq 0$. This leads to a contradiction. Thus we have

Theorem 7. *Let $f(z)$ be uniform and meromorphic in D. Suppose that H contains no interior point and the number $p=\max p_i$ is finite. Then $f(z)$ is p-valent in D.*

In connection with theorem 6, the following example is somewhat interesting. Consider an integral function $w(z)=e^{-z^2}\cos z$. Take as D an angular domain: $|\arg z| < \frac{\pi}{4}-\delta$ $(\delta>0)$. Since $w(z)\to 0$ uniformly as $z\to\infty$ in D, the set H is identical with the image of both sides of D and Γ contains only one point, that is, $w=0$, for $w(z)$ vanishes at $z=\frac{(2n+1)\pi}{2}$. It is clear that H never contains any interior point.