52. *Theory of Connections in a Kawaguchi Space of Order Two.*

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In the present paper the writer will give the foundation to the geometry in a Kawaguchi space of order two and of dimension n by introducing connections under the point transformation group. An element of this space is not a point but a line element of the third order. In the space of dimension two E. Cartan has discussed already the theory of invariants under the contact transformation group.[1]

1. The metrics in the space with a point coordinate system x^i $(i=1, 2, \ldots\ldots, n)$ is given by $s=\int F(x, x', x'')dt$, putting $x'^i=\frac{dx^i}{dt}$ and $x''^i=\frac{d^2x^i}{dt^2}$. When this metrics varies by a change of parameter t the theory of connections can be easily and completely discussed. We shall assume therefore that this metrics is invariant under any change of parameter. This assumption leads to the necessary and sufficient conditions:

$$(1) \qquad F_{(2)i}x'^i=0\,, \qquad 2F_{(2)i}x''^i+F_{(1)i}x'^i=F\,.^{2)}$$

The Synge vectors $\overset{a}{E}_i\ (a=0, 1, 2)$ are not invariant under a change of parameter, but it is not difficult to find out the invariant vectors which have the forms

$$(2) \qquad \overset{0}{\mathfrak{E}}_i=F^{-1}\overset{0}{E}_i\,, \qquad \overset{1}{\mathfrak{E}}_i=\overset{1}{E}_i+F^{-1}F^{(1)}\overset{2}{E}_i\,, \qquad \overset{2}{\mathfrak{E}}_i=F\overset{2}{E}_i\,.$$

The tensor

$$(3) \qquad g_{ij}=2F^3F_{(2)i(2)j}+\overset{1}{\mathfrak{E}}_i\overset{1}{\mathfrak{E}}_j+\overset{2}{\mathfrak{E}}_i\overset{2}{\mathfrak{E}}_j$$

is also invariant and its determinant is not equal to zero, if the matrix $(2F_{(2)i(2)j}+\overset{2}{\mathfrak{E}}_i\overset{2}{\mathfrak{E}}_j)$ is of rank $n-1$, as follows from (1): $g_{ij}x'^j=-F\overset{1}{\mathfrak{E}}_i$. g_{ij} may be functions of a line element of the third order. Now we shall take this tensor as the fundamental tensor in our space and its contravariant components are defined by $g_{ij}g^{jk}=\delta_i^k$. It can be seen without difficulty that $F^3\overset{1}{\mathfrak{E}}_{i(3)j}=g_{ij}-\overset{1}{\mathfrak{E}}_i\overset{1}{\mathfrak{E}}_j$.

2. We call a quantity Q (scalar, vector, tensor, etc.) which behaves under a change of parameter $t^*=t^*(t)$ in the way:

$$Q(t^*)=a^pQ(t)\,, \qquad a=\frac{dt}{dt^*}\,,$$

1) E. Cartan, Journal de Mathématique, (9) **15** (1936), pp. 42–69.

2) We adopt here the same notations as in the previous paper of the present author: Some intrinsic derivations in a generalized space, Proc. **12** (1936), pp. 149–151.

a geometrical quantity of class p and order m, when the quantity Q contains the derivatives of coordinates to the mth order. Especially a geometrical quantity of class 0 is named *an intrinsic quantity.* F is a geometrical scalar of class 1 and order 2, while $\overset{0}{\mathfrak{E}}_i$, $\overset{1}{\mathfrak{E}}_i$ and $\overset{2}{\mathfrak{E}}_i$ are intrinsic vectors of orders 4, 3 and 2 resp. The tensor g_{ij} is also intrinsic of order 3.

The scalar $\Phi=\dfrac{d}{dt}\log F$ of order 3 is transformed by a change of parameter in the way

$$\Phi(t^*)=\alpha\Phi(t)+\frac{1}{\alpha}\frac{d\alpha}{dt^*}.$$

Under the assumption that $\sigma\equiv g^{ij}\overset{2}{\mathfrak{E}}_i\overset{2}{\mathfrak{E}}_j \neq 1$,[1] we have a scalar $\Psi=F^{-1}F^{(2)}-2F^2(1-\sigma)^{-1}g^{ij}\overset{2}{\mathfrak{E}}_i\overset{0}{\mathfrak{E}}_j$ of order 3, for which it holds good

$$\Psi(t^*)=\alpha^2\Psi(t)+3\frac{d\alpha}{dt^*}\Phi(t)+\frac{1}{\alpha}\frac{d^2\alpha}{dt^{*2}}.$$

3. It is known after some calculations that

$$F^3\overset{1}{D}_j(F_{(2)i})X^j=2F^3F_{(2)i(2)j}\frac{dX^j}{dt}+F^3F_{(2)i(1)j}X^j+F^3\Phi F_{(2)i(2)j}X^j,$$

$$\frac{1}{2}F^2F_{(2)i}\overset{1}{D}_j(F)X^j=F^2F_{(2)i}F_{(2)j}\frac{dX^j}{dt}+\frac{1}{2}F^2F_{(2)i}F_{(1)j}X^j+\frac{1}{2}F^2\Phi F_{(2)i}F_{(2)j}X^j,$$

$$\overset{1}{\mathfrak{E}}_i\left\{(\overset{1}{\mathfrak{E}}_jX^j)^{(1)}-\left(2\overset{0}{E}_j+\frac{2}{1-\sigma}g^{kl}\overset{2}{\mathfrak{E}}_k\overset{0}{E}_l\overset{2}{\mathfrak{E}}_j\right)X^j\right\}=\overset{1}{\mathfrak{E}}_i\overset{1}{\mathfrak{E}}_j\frac{dX^j}{dt}-\overset{1}{\mathfrak{E}}_i\Xi_jX^j$$

are all geometrical vectors of class 1 and order 3, where Ξ_j are functions of a line element of the third order and X^i is an arbitrary intrinsic vector. From these vectors there can be derived a covariant differentiation of X^i, which is a geometrical vector of class 1 and order 3:

$$\frac{\delta X^i}{dt}=\frac{dX^i}{dt}+\Gamma^i_jX^j, \tag{4}$$

where

$$\Gamma^i_j=g^{ik}\left(F^3F_{(2)k(1)j}+\frac{1}{2}F^2F_{(2)k}F_{(1)j}\right)+\frac{1}{2}\Phi\left(\delta^i_j+\frac{x'^i}{F}\overset{1}{\mathfrak{E}}_j\right)+\frac{x'^i}{F}\Xi_j$$

is a geometrical quantity (not tensor) of class 1 and order 3. As the transformation law of Γ^i_j by a coordinate transformation is

$$\Gamma'^\lambda_\mu=\Gamma^i_j\frac{\partial x'^\lambda}{\partial x^i}\frac{\partial x^j}{\partial x'^\mu}-\frac{\partial^2x'^\lambda}{\partial x^i\partial x^j}\frac{\partial x^i}{\partial x'^\mu}x'^j. \tag{5}$$

Owing to this relation

$$\overset{2}{D}\Gamma^i_j\equiv3\Gamma^i_{j(3)k}dx''^k+2\Gamma^i_{j(2)k}dx'^k+\Gamma^i_{j(1)k}dx^k$$

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1) When the matrix $(F_{(2)i(2)j})$ is of rank $n-1$, we have also a similar scalar, even if $\sigma=1$.

is transformed in the way

$$\overset{2}{D}\Gamma^{\lambda}_{\mu} = \overset{2}{D}\Gamma^i_j \frac{\partial x^{\lambda}}{\partial x^i}\frac{\partial x^j}{\partial x^{\mu}} - \frac{\partial^2 x^{\lambda}}{\partial x^i \partial x^j}\frac{\partial x^i}{\partial x^{\mu}} dx^j$$

and

$$\overset{1}{D}\Gamma^i_j \equiv 3\Gamma^i_{j(3)k} dx'^k + \Gamma^i_{j(2)k} dx^k , \qquad \overset{0}{D}\Gamma^i_j \equiv \Gamma^i_{j(3)k} dx^k$$

are both tensors. It can be seen that the quantity

$$\overset{2}{D}\Gamma^i_j + \Phi \overset{1}{D}\Gamma^i_j + \Psi \overset{2}{D}\Gamma^i_j \equiv \sum_{a=0}^{2} \overset{a}{\Gamma}{}^i_{jk} dx^{(a)k} \tag{6}$$

is an intrinsic quantity of order 3 and is transformed just as $\overset{2}{D}\Gamma^i_j$, from which an intrinsic covariant differential of an intrinsic vector X^i of order 3 follows immediately:

$$\delta X^i = dX^i + \sum_{a=0}^{2} \overset{a}{\Gamma}{}^i_{jk} X^j dx^{(a)k} .\text{[1]} \tag{7}$$

From this differential one can define a covariant differential of an arbitrary tensor by the usual method. If v^i is a geometric tensor of class p, the covariant differential of v^i is defined by

$$\delta v^i = dv^i + \sum_{a=0}^{2} \overset{a}{\Gamma}{}^i_{jk} v^j dx^{(a)k} - p v^i d \log F . \tag{8}$$

4. We must define the base connections, for that the covariant derivatives and curvature tensors should be derived. We have as the base connections

$$F^3 g^{ij} \delta \overset{1}{\mathfrak{E}}_i \equiv \delta x'''^j = \left(\delta^j_i + \overset{1}{\mathfrak{E}}_i \frac{x'^j}{F}\right) dx'''^i + \sum_{a=0}^{2} \underset{3}{\overset{a}{\Lambda}}{}^j_i dx^{(a)i} ,$$

$$\frac{1}{3} F^3 g^{ij} \overset{2}{D} \overset{1}{\mathfrak{E}}_i \equiv \delta x''^j = \left(\delta^j_i + \overset{1}{\mathfrak{E}}_i \frac{x'^j}{F}\right) dx''^i + \sum_{a=0}^{1} \underset{2}{\overset{a}{\Lambda}}{}^j_i dx^{(a)i} ,$$

$$\delta x'^i = dx'^i + \underset{1}{\overset{0}{\Lambda}}{}^i_j dx^j \qquad (\underset{1}{\overset{0}{\Lambda}}{}^i_j = \Gamma^i_j)$$

and the following relation can be verified easily

$$\delta X^i = \sum_{a=0}^{3} \nabla^{(a)}_j X^i \cdot \delta x^{(a)j} ,$$

where

$$\nabla^{(3)}_j X^i = X^i_{(3)j} ,$$

$$\nabla^{(2)}_j X^i = X^i_{(2)j} - \nabla^{(3)}_k X^i \cdot \underset{3}{\overset{2}{\Lambda}}{}^k_j + \overset{2}{\Gamma}{}^i_{kj} X^k ,$$

$$\nabla^{(1)}_j X^i = X^i_{(1)j} - \nabla^{(3)}_k X^i \cdot \underset{3}{\overset{1}{\Lambda}}{}^k_j - \nabla^{(2)}_k X^i \cdot \underset{2}{\overset{1}{\Lambda}}{}^k_j + \overset{1}{\Gamma}{}^i_{kj} X^k ,$$

1) This connection is not metric, i. e. $\delta g_{ij} \neq 0$, but it is easy to derive a metric connection from this. See A. Kawaguchi: Beziehung zwischen der metrischen linearen Übertragung und einer nicht-metrischen in einem allgemeinen metrischen Raume, which will be published soon in Proceedings Kon. Akad. v. Wetensch. Amsterdam.

$$\nabla_j^{(0)} X^i = X^i_{(0)j} - \nabla_k^{(3)} X^i \cdot \overset{0}{\underset{3}{\Lambda}}{}^k_j - \nabla_k^{(2)} X^i \cdot \overset{0}{\underset{2}{\Lambda}}{}^k_j - \nabla_k^{(1)} X^i \cdot \overset{0}{\underset{1}{\Lambda}}{}^k_j + \overset{0}{\Gamma}{}^i_{kj} X^k$$

are the covariant derivatives of X^i. The class of the geometric tensor $\nabla_j^{(a)} X^i$ is a.

The curvature and torsion tensors are calculated from Γ's and Λ's and the fundamental theorems can be proved by a similar method as in the case of an affine connection.

5. In conclusion we shall add a remark on a Kawaguchi space of order m. From the vectors

$$m\overset{1}{D}_j(F_{(m)i}) X^j = mF_{(m)i(m)j}\frac{dX^j}{dt} + F_{(m)i(m-1)j} X^j ,$$

$$\overset{1}{E}_i \left\{ (\overset{1}{E}_j X^j)^{(1)} - m\overset{0}{E}_j X^j \right\} = \overset{1}{E}_i \overset{1}{E}_j \frac{dX^j}{dt} + \overset{1}{E}_i X^j \sum_{a=0}^{m-1} (-1)^a (a-m) F_{(a)j}{}^{(a)} ,$$

we get by the same method as above stated the covariant differential of a vector X^i:

$$\delta X^i = dX^i + \sum_{a=1}^{2m-1} a\Gamma^i_{j(a)k} X^j dx^{(a-1)k} ,$$

where

$$\Gamma^i_j = \frac{F^{2m-1}}{m} g^{ik} F_{(m)k(m-1)j} + \frac{x'^i}{F} \sum_{a=0}^{m-1} (-1)^a (a-m) F_{(a)j}{}^{(a)} ,$$

$$g_{ij} = F^{2m-1} F_{(m)i(m)j} + \overset{1}{E}_i \overset{1}{E}_j , \qquad g^{ij} g_{jk} = \delta^i_k , \qquad | g_{jk} | \neq 0 ,$$

$$\overset{1}{E}_i = \sum_{\beta=1}^{m} (-1)^m \beta F_{(\beta)i}{}^{(\beta-1)} , \qquad \overset{0}{E}_i = \sum_{\beta=0}^{m} (-1)^m F_{(\beta)i}{}^{(\beta)} .$$

But this differential is not geometric, even if X^i is geometric.