103. *Notes on Fourier Series (II): Convergence Factor.*

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(Comm. by M. FUJIWARA, M.I.A., Dec. 13, 1937.)

1. R. Salem has proved the following theorem:[1]

If $f(x)$ is a continuous function with period 2π and its Fourier coefficients be a_n and b_n, then the relation

$$\lim_{s\to 0}\left\{\frac{1}{2}a_0+\sum_{n=1}^{\infty}\frac{a_n\cos nx+b_n\sin nx}{1+s\log n}\right\}=f(x) \tag{1}$$

holds good almost everywhere.

In this relation we must notice that the series in the bracket of the right hand side is convergent for every positive value of s and for almost all x.

One of the present authors[2] has proved that

$$\lim_{s\to 0}\left\{\frac{1}{2}a_0+\sum_{n=1}^{\infty}\frac{a_n\cos nx+b_n\sin nx}{1+s\sqrt{\log n}}\right\}=f(x) \tag{2}$$

almost everywhere for squarely integrable function $f(x)$.

The object of this paper is to prove that (1) is true for any integrable function and there is the corresponding relation for the function in L^p $(1\leqq p\leqq 2)$.

2. *Theorem. If $f(x)\varepsilon L^p(1\leqq p\leqq 2)$ and is periodic with period 2π and a_n and b_n are its Fourier coefficients, then we have*

$$\lim_{s\to 0}\left\{\frac{1}{2}a_0+\sum_{n=1}^{\infty}\frac{a_n\cos nx+b_n\sin nx}{1+s(\log n)^{1/p}}\right\}=f(x) \tag{3}$$

almost everywhere.

Actually we can replace the factors $\left\{\frac{1}{1+s(\log n)^{1/p}}\right\}$ by the more general sequence $\{\psi_n(s)\}$ which satisfies certain conditions.[3] But we make here no attention to this.

For the proof we make use of the theorem:

Lemma. If $f(x)\varepsilon L^p$ $(1\leqq p\leqq 2)$, then

$$\sum_{n=1}^{\infty}\frac{a_n\cos nx+b_n\sin nx}{(\log n)^{1/p}}$$

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1) R. Salem, Comptes Rendus, **205** (1937), pp. 14–16, **205** (1937), pp. 311–313. In the latter paper, Salem remarked that more generally for a bounded function, (2) holds good.

2) T. Kawata, Proc. **13** (1937), 381–384.

3) Cf. Salem, loc. cit. and T. Kawata, loc. cit.

converges for x in a set E with measure 2π. And for every x in E the n-th partial sum of the series (3) is $o\big((\log n)^{1/p}\big)$.

The case $p=1$ is due to Hardy-Littlewood-Plessner, the case $p=2$ due to Kolmogoroff-Seliverstoff-Plessner and the remaining case was recently proved by Littlewood-Paley.[1]

From Lemma we can easily verify that the series in the left hand side of (3) converges in E for all s (>0) and the n-th partial sum is $o\big((\log n)^{1/p}\big)$.

We will prove the theorem for the case $p=1$. The other case can be proved quite similarly.

3. Let us put

$$f(x, s)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}\frac{a_n\cos nx+b_n\sin nx}{1+s\log n} \tag{4}$$

which converges almost everywhere by Lemma.

By the twice application of the Abel's lemma and by Lemma, we have

$$f(x, s)=\lim_{N\to\infty}\left\{\frac{1}{2}a_0+\sum_{n=1}^{\infty}\frac{a_n\cos nx+b_n\sin nx}{1+s\log n}\right\}$$

$$=\sum_{n=0}^{\infty}K_n(x)\,\Delta^2\left(\frac{1}{1+s\log n}\right),^{2)}$$

where

$$K_0(x)=\frac{1}{2}a_0,$$

$$K_n(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x+t)\frac{\sin^2\frac{nt}{2}}{\sin^2\frac{t}{2}}dt \qquad (n>1),$$

$$\Delta a_p=a_p-a_{p+1}, \qquad \Delta^2 a_p=\Delta(\Delta a_p).$$

We will suppose that $f(x)\geqq 0$. Since $K_n(x)\geqq 0$ and $\left\{\frac{1}{1+s\log n}\right\}$ is a convex sequence, $f(x,s)\geqq 0$. We have

$$\int_{-\pi}^{\pi}\overline{\lim_{s\to 0}}f(x,s)\,dx\leqq\int_{-\pi}^{\pi}\lim_{\sigma\to 0}\,[\underset{\sigma\leq s\leq 1}{\text{l. u. b.}}\,f(x,s)]\,dx$$

$$\leqq\underline{\lim_{\sigma\to 0}}\int_{-\pi}^{\pi}\underset{\sigma\leq s\leq 1}{\text{l. u. b.}}\,f(x,s)\,dx$$

$$=\underline{\lim_{\sigma\to 0}}\int_{-\pi}^{\pi}\underset{\sigma\leq s\leq 1}{\text{l. u. b.}}\left[\sum_{n=0}^{\infty}K_n(x)\,\Delta^2\left(\frac{1}{1+s\log n}\right)\right]dx.$$

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1) See Zygmund, Trigonometrical series, 1935. pp. 58–59, pp. 252–255.

2) We must replace $\frac{1}{1+s\log n}$ by 1, if $n=0$.

Now we can find n_s such that $\Delta^2\left(\frac{1}{1+s\log x}\right)$ is decreasing for $x \geqq n_s$ and is increasing for $x \leqq n_s$. Thus we have

$$(5)\qquad \int_{-\pi}^{\pi} \overline{\lim_{s\to 0}} f(x,s)\,dx \leqq \overline{\lim_{\sigma\to 0}} \int_{-\pi}^{\pi} \underset{\sigma\leqq s\leqq 1}{\text{l. u. b.}} \Big[\sum_{\nu=0}^{n_s-3} K_n(x)\Delta^2\left(\frac{1}{1+s\log n}\right) + \sum_{\nu=n_s-2}^{n_s+2} K_n(x)\Delta^2\left(\frac{1}{1+s\log n}\right) + \sum_{\nu=n_s+3}^{\infty} K_n(x)\Delta^2\left(\frac{1}{1+\sigma\log n}\right)\Big] dx$$

$$\leqq \overline{\lim_{\sigma\to 0}} \int_{-\pi}^{\pi} \Big[f(x,1)+f(x,\sigma) + \underset{\sigma\leqq s\leqq 1}{\text{l. u. b.}} \sum_{n_s-2}^{n_s+2} K_n(x)\Delta^2\left(\frac{1}{1+s\log n}\right)\Big] dx$$

$$\leqq c_1 \int_0^{2\pi} f(x)\,dx .$$

4. Let us consider the general integrable function $f(x)$. Let us put

$$f(x)=f^+(x)-f^-(x),$$

where $\qquad f^+(x)=f(x), \quad f^-(x)=0, \quad \text{if } f(x)\geqq 0;$

$\qquad\qquad =0 \quad , \qquad\quad =f(x), \quad \text{if } f(x)<0.$

Then $|f|\geqq f^+\geqq 0$ and $|f|\geqq f^-\geqq 0$. Thus we get from (5)

$$(6)\qquad \int_{-\pi}^{\pi} \overline{\lim_{s\to 0}} |f(x,s)|\,dx \leqq \int_{-\pi}^{\pi} \{\overline{\lim_{s\to 0}} f^+(x,s) + \overline{\lim_{s\to 0}} f^-(x,s)\}\,dx$$

$$\leqq c_2 \int_{-\pi}^{\pi} |f(x)|\,dx ,$$

where $f^+(x,s)$ and $f^-(x,s)$ represent the series in left hand side of (4), constructed from $f^-(x)$ instead of $f(x)$.

Now let the Fejér sum of the Fourier series of $f(x)$ be $\sigma_n(x)$ and form $\sigma_n(x,s)$ from $\sigma_n(x)$ as before. Then we have by (6)

$$\int_{-\pi}^{\pi} \overline{\lim_{s\to 0}} |f(x,s)-\sigma_n(x,s)|\,dx \leqq c_3 \int_{-\pi}^{\pi} |f(x)-\sigma_n(x)|\,dx ,$$

which tends to zero as $n\to\infty$. Thus the known result concerning the mean convergence shows that there exist a sequence of integers $\{n_k\}$ and a set F such that $mF=2\pi$ and

$$(7)\qquad \lim_{n_k\to\infty} \overline{\lim_{s\to 0}} |f(x,s)-\sigma_{n_k}(x,s)|=0$$

for x in F.

5. We have

$$\lim_{s,\, s'\to 0} |f(x, s)-f(x, s')| \leqq \overline{\lim_{s\to 0}}\, |f(x, s)-\sigma_{n_k}(x, s)| + \overline{\lim_{s'\to 0}}\, |f(x, s')-\sigma_{n_k}(x, s')|\,.$$

Letting $n_k \to \infty$, we reach the result that

$$\lim_{s\to 0} f(x, s)$$

exists almost everywhere. The fact that the limit function $g(x)$ is equal to $f(x)$, is immediate. For (7) yields us

$$\lim_{n_k\to\infty} |g(x)-\sigma_{n_k}(x)|=0\,.$$

Since $\sigma_{n_k}(x)$ tends to $f(x)$ almost everywhere, $g(x)=f(x)$ almost everywhere. Thus the theorem is proved.