PAPERS COMMUNICATED

1. *An Invariant Property of Siegel's Modular Function.*

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C. L. Siegel[1] recently defined the following remarkable function

$$f_r(X)=\sum_{(P,Q)}|PX+Q|^{-2r},\text{[2]}$$

where X is a quadratic matrix of the dimension n with a positive "imaginary part" and P and Q are matrices of the same dimension having rational integral components, while $\sum$ sums over all non-associated symmetrical pairs of matrices P and Q without a left common divisor.

It is absolutely and uniformly convergent when an integer $r>\frac{n(n+1)}{2}$ and represents a modular function of the nth. degree and of the dimension $-2r$.

In making use of the system of representatives of the classes of transformations of Siegel's modular group, that I have given in my former paper,[3] I will extend in this work a property of Eisentein's series, due to Mr. Hecke,[4] to this new function: namely I prove the following

Theorem: Let $T_i=\begin{pmatrix}A_i & B_i\\ 0 & D_i\end{pmatrix}$, $i=1, 2, \ldots\ldots, k$ be the complete system of representatives of the classes of transformations of the degree m, then by the linear operator $\sum_{i=1}^{k}T_i|D_i|^{-2r}$ the function $f_r(X)$ is multiplied by a constant factor N;

$$\sum_i|D_i|^{-2r}f_r\big(T_i(X)\big)=Nf_r(X)\,.$$

Firstly I prove the

Lemma: The number of the classes of transformations of the degree m, $T=\begin{pmatrix}A & B\\ C & D\end{pmatrix}$, in which A and B are two given matrices, depends only on the common divisor G which makes two matrices $G^{-1}A$ and $G^{-1}B$ left-relatively-prime.

Proof: As T is a transformation of the degree m, namely

1) C. L. Siegel, Analytische Theorie der quadratischen Formen, 1.
2) $|PX+Q|$ represents the determinant of the matrix $PX+Q$.
3) M. Sugawara. On the transformation theory of Siegel's modular group.
4) E. Hecke. Die Prinzahlen in der Theorie der elliptischen Modulfunktionen.

$$(1) \qquad T'JT = mJ\,, \quad \text{where} \quad J = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix},$$

we have $T^{-1}JT'^{-1} = \frac{1}{m}J$ and thus $TJT' = mJ$, so that the condition (1) is equivalent to the condition

$$(1') \qquad AD' - BC' = mE\,, \qquad AB' = BA', \qquad CD' = DC'.$$

Because a left common divisor of A and B is a divisor of mE, there exists always such a non-singular matrix G with rational integral components as mentioned in the enunciation of the lemma.

Put $A_1 = G^{-1}A$, $B_1 = G^{-1}B$ and $K = G^{-1}m$. Then (1′) is again equivalent to

$$(2) \qquad A_1D' - B_1C' = K\,, \qquad A_1B_1' = B_1A_1'\,, \qquad CD' = DC'.$$

Let C_0, D_0 be a particular solution of (1′) (or (2)), then by Siegel's lemma 42, *l. c.* the general solution C, D of (1′) can be represented in the form

$$C = C_0 + SA_1\,, \qquad D = D_0 + SB_1\,,$$

where S is a symmetrical matrix with rational integral components. Thus the general transformation $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ of the degree m having the given matrices A, B in the "first row" is obtained in the form $\begin{pmatrix} E & 0 \\ SG^{-1} & E \end{pmatrix}\begin{pmatrix} A & B \\ C_0 & D_0 \end{pmatrix}$. Two transformations given in this form belong to the same class of transformations if and only if the difference of the corresponding matrices S_1G^{-1} and S_2G^{-1} is a symmetric matrix with rational integral components. Let us therefore call two matrices S_1G^{-1}, S_2G^{-1}, where S_1 and S_2 are two symmetric matrices with rational integral components, *equivalent* when they differ only by a symmetric matrix with rational integral components $S^{(1)}$, namely $S_2G^{-1} = S_1G^{-1} + S^{(1)}$. Equivalent matrices form a class. Let $n(G)$ be the number of the classes of such matrices, then the required number of the classes of transformations in the lemma is $n(G)$.

Proof of the theorem: By Siegel's lemma 42, *l. c.* we can complete P, Q to a modular substitution $M = \begin{pmatrix} P & Q \\ U & V \end{pmatrix}$, then $T = MT_i = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is a transformation of the degree m belonging to the same class as T_i and we have $|D_i|\,|PT_i(X) + Q| = |AX + B|$. Conversely let T be any transformation of the degree m and T_i be the representative of the class of T, then $MT_i = T$, where $M = \begin{pmatrix} P & Q \\ U & V \end{pmatrix}$ is a modular substitution. If A, B and T_i or the class of T are given here, P and Q are uniquely determined from this relation, as $PA_i = A$, $PB_i + QD_i = B$. By a fixed T_i there exists one to one correspondence between non-associated pairs of matrices P, Q and A, B. Therefore we have

$$\sum_{i=1}^{k} |D_i|^{-2r} f_r\big(T_i(X)\big) = \sum{}' \frac{1}{|AX+B|^{2r}}\,,$$

where in the sum on the right appear all the non-associated pairs of matrices which make the 1st. rows of the transformations of the degree m and each row A, B as often as there exist transformations of the form $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ belonging to different classes of transformations, *i. e.* $n(G)$ times, if G is a "greatest common left divisor" of A, B.

Let us first sum up such terms in this series that have A, B with a determinate greatest common left divisor G and then let G run over the "left non-associated divisors" of mE. Thus we get

$$\sum_{i=1}^{k} |D_i|^{-2r} f_r\big(T_i(X)\big) = N f_r(X),$$

where $N = \sum_{G|mE} \frac{n(G)}{|G|^{2r}}$, in which G runs over left-non-associated divisor of mE.

It is already known by Siegel, *l. c.*, that the constant term of the Fourier expansion of the function $f_r(X)$ is not zero; so that the system of all modular forms of the *stufe* 1 and of the dimension $-2r$ decomposes into two parts: one part is produced by $f_r(X)$ and the other by modular forms whose constant terms of the Fourier expansions are zero and each part is transformed by the linear operator $\sum_{i=1}^{k} T_i |D_i|^{-2r}$ into itself.[1] Therefore all the other proper modular forms of the linear operator belong to the second part. Hence we get the following important

Theorem: Siegel's function $f_r(X)$ is characterified, except a constant factor, among the modular forms of the degree n, of the *stufe* 1 and of the dimension $-2r$, as that proper function of the linear operator $\sum_{i=1}^{k} T_i |D_i|^{-2r}$, whose constant term in the Fourier expansion is not zero.

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1) Because $A_i X D_i^{-1}$, $i=1, 2, \ldots\ldots, k$, arc symmetric matrics and have with X also *positive* imaginary parts.