# *13.* On Siegel's Modular Function of the Higher Stufe.

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In this note we are concerned with modular functions of the degree $n$, of the dimension $-2r$ and of the *stufe* $m$, which is an extension of Eisenstein's series of the *stufe* $m$, due to Mr. Hecke,[1] to the case of the degree $n$, and deduce some of the corresponding properties.

We call Siegel's modular function of the degree $n$, of the dimension $-2r$, and of the *stufe* $m$ the following function,

$$f_r(X; P_1, Q_1; m) = \sum_{\substack{P\equiv P_1 \\ Q\equiv Q_1 \mod m \\ (P,Q)_m}} \frac{1}{|PX+Q|^{2r}},^{2)}$$

where $X$ is a symmetric matrix with a positive "imaginary part" and $P_1$, $Q_1$ form a given symmetrical pair of matrices with rational integral components and have no left common divisor, while $\sum$ sums over mod $m$ non-associated symmetrical pair of matrices $P$ and $Q$ which are congruent to $P_1$ and $Q_1$ respectively and have no left common divisor.

Here we call two symmetrical pairs of matrices, $P$, $Q$ and $P_0$, $Q_0$ "associated mod $m$" when there exists an unimodular matrix $U$, congruent to $\pm E$ mod $m$, such that the relations $P_0 = UP$, $Q_0 = UQ$ hold.

As in the case of Siegel's modular function of the 1st. stufe, it is absolutely and uniformly convergent when the integer $r > \frac{n(n+1)}{2}$ and represents an analytic function of $X$ in the domain $H$ in which $X$ has a positive imaginary part.

The behavior under a modular substitution $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is as follows. Let us complete $P$, $Q$ to a modular substitution $\begin{pmatrix} P & Q \\ U & V \end{pmatrix}$, then

$$\begin{pmatrix} P & Q \\ U & V \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} PA+QC & PB+QD \\ UA+VC & UB+VD \end{pmatrix}$$

is also a modular substitution, so that $K = PA+QC$ and $L = PB+QD$ form a symmetrical pair of matrices without a left common divisor, and

$$K \equiv K_1 = P_1A + Q_1C,$$

$$L \equiv L_1 = P_1B + Q_1D \qquad \mod m.$$

1) E. Hecke. Theorie der Eisensteinschen Reihen höherer Stufe and ihre Anwendung auf Funktionentheorie und Arithmetik.

2) Capital letters represent $n$-dimensional matrices, while small letters represent integers.

Hence from

$$f_r\big((AX+B)(CX+D)^{-1}; P_1, Q_1; m\big)$$
$$=|CX+D|^{2r}\sum_{\substack{P\equiv P_1\\ Q\equiv Q_1 \bmod m\\ (P,Q)_m}}\frac{1}{|(PA+QC)X+PB+QD|^{2r}}$$
$$=|CX+D|^{2r}\sum_{\substack{K\equiv K_1\\ L\equiv L_1 \bmod m\\ (K,L)_m}}\frac{1}{|KX+L|^{2r}}=|CX+D|^{2r}f_r(X; K_1, L_1; m)$$

we get

$$(1)\qquad |CX+D|^{2r}f_r(X; K_1, L_1; m)$$
$$=f_r\big((AX+B)(CX+D)^{-1}; K_1D'-L_1C', -K_1B'+L_1A'; m\big).$$

Let $M$ be a substitution of the principal congruence group mod $m$ $\Gamma(m)$, then (1) becomes

$$(2)\quad f_r\big((AX+B)(CX+D)^{-1}; P_1, Q_1; m\big)=|CX+D|^{2r}f_r(X; P_1, Q_1; m).$$

Especially it is an absolute invariant by the modular substitution $\begin{pmatrix}E & mS\\ 0 & E\end{pmatrix}$, where $S$ is a symmetric matrix with rational integral components, so that it can be expanded into Fourier series

$$(3)\qquad \sum_I a(I)e^{\frac{2\pi i}{m}\sigma(IX)},$$ [1]

where $\sum$ sums over all integral form $x'Ix$ ($x$ is a $n$-dimensional vector), but by the same reason as in the case of the 1st stufe $a(I)=0$ for all $I$ for which $x'Ix$ can take also negative values in real $x$. Thus $\sum$ may sum only over non-negative forms $x'Ix$

$$\sum_{I\geq 0} a(I)e^{\frac{2\pi i}{m}\sigma(IX)}.$$

It follows from here that the function $f_r(X; P_1, Q_1; m)$ is a modular form of the degree $n$, of the dimension $-2r$, and of the stufe $m$.

As the explicit form of the Fourier expansion of the function is complicated, we get its constant term in the following way.

Let $Y$ and $Z$ be real resp. imaginary part of $X$, $X=Y+iZ$, and take for $Z$ the positive diagonalform $zE$ ($z\to\infty$), then $a(I)e^{\frac{2\pi i}{m}\sigma(IX)}\to 0$ for $I\neq 0$ and by suitable choice of $Y$ $|PX+Q|^{-2r}\to 0$ for $P\neq 0$.

The 1st part follows at once from the fact that the non-negative form whose diagonal components are all zero is the form 0.

For the proof of the 2nd part put $|PX+Q|^{-2r}=|P_0|^{-2r}|R'XR+P_0^{-1}Q_0|^{-2r}$,

where $P=U_1\begin{pmatrix}P_0^{(r)} & 0\\ 0 & 0\end{pmatrix}U'$, $Q=U_1\begin{pmatrix}Q_0^{(r)} & 0\\ 0 & E^{(n-r)}\end{pmatrix}U^{-1}$, $U=(R^{(n,r)}\ C_0)$,

1) $\sigma(A)$ represents the trace of the matrix $A$.

$|U|=|U_1|=1$, $|P_0|>0$ and $P_0$, $Q_0$ form a symmetrical pair of matrices,[1] and $R'ZR=F'F$, $R'YR+P_0^{-1}Q_0=F'D_0F$ with a real matrix $F=F^{(r)}$ and a real diagonal matrix $D_0=D_0^{(r)}$,

then $$|R'XR+P_0^{-1}Q_0|=|R'ZR|\,|D_0+iE|\,.$$

If we take instead of $R$ its associated matrix, $R'ZR$ become another representative of that class. So we can assume that the definite quadratic form $R'ZR$ is reduced in the meaning of Hermite. Then it follows from the Hermitian condition of reducibility that the quotient of the product of the diagonal elements of the matrix $R'ZR$ and the determinant $|R'ZR|$ is bounded by a constant independent of $R$. Therefore when $z\to\infty$, the product of the diagonal elements of $R'ZR$, hence also $|R'ZR|$ become infinity. If $Y$ is so chosen that $|D_0+iE|\neq 0$, we have $|PX+Q|^{-2r}\to 0$.

Thus the constant term of its Fourier expansion is

$$(4)\qquad \delta(P_1,Q_1;m)=\begin{cases}1, & \text{when } P_1\equiv 0,\ Q_1\equiv U \bmod m, \text{ where } U \text{ is unimodular.}\\ 0, & \text{in all other cases.}\end{cases}$$

In the following investigation it is convenient to use a "homogeneous coordinate" $P$, $Q$ of $X$ and the words in the homogeneous form.[2]

We call a class of symmetrical pairs of matrices $P$, $Q$ without a left common divisor a "rational point," namely two symmetrical pairs of matrices without a left common divisor $P$, $Q$ and $P_0$, $Q_0$ represent the same rational point if and only if $P=UP_0$, $Q=UQ_0$, where $U$ represents a unimodular matrix with rational integral components.

Two rational points $(P, Q)$ and $(P_0, Q_0)$ are said to be equivalent by the principal congruence group mod $m$ $\Gamma(m)$ if and only if there exists a substitution $M^{(m)}$ of the group $\Gamma(m)$ such that the relation

$$(5)\qquad (P,Q)\,M^{(m)}=(UP_0,\ UQ_0)$$

holds.

This condition is equivalent to

$$(5')\qquad P\equiv UP_0\,,\qquad Q\equiv UQ_0 \qquad \bmod m\,.$$

The condition (5′) follows evidently from (5). Assume that (5′) holds. Let us complete $P$, $Q$ and $UP_0$, $UQ_0$ to modular substitutions $M=\begin{pmatrix}P & Q\\ V & W\end{pmatrix}$, $M_0=\begin{pmatrix}UP_0, & UQ_0\\ V_0, & W_0\end{pmatrix}$ resp., then there exists a modular substitution $N=\begin{pmatrix}A & B\\ C & D\end{pmatrix}$ such that $N=M_0M^{-1}=\begin{pmatrix}U(P_0W'-Q_0V') & U(-P_0Q'+Q_0P')\\ V_0W'-W_0V' & -V_0Q'+W_0P'\end{pmatrix}$. From (5′), $PW'-QV'=E$, $PQ'=QP'$, we get $A\equiv E$, $B\equiv 0 \bmod m$, and then from $A'D-C'B=E$, $A'C=C'A$ we get $D\equiv E$, $C\equiv C_0 \bmod m$, where $C_0$ is a symmetric matrix; so that $N\equiv\begin{pmatrix}E & 0\\ C_0 & E\end{pmatrix} \bmod m$. Multi-

1) L. Siegel: Analytische Theorie der quadratischen Formen, 1. lemma 42.

2) For a moment we define only "rational points" in homogeneous form.

$$X=P^{-1}Q\,,\quad \text{when}\quad |P|\neq 0\,.$$

plying a substitution $N_1=\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} E & 0 \\ -C_0 & E \end{pmatrix}$ of the group $\Gamma(m)$ to the modular substitution $\begin{pmatrix} P & Q \\ C_0P+V & C_0Q+W \end{pmatrix}$ we get the substitution $\begin{pmatrix} UP_0 & UQ_0 \\ V_0 & W_0 \end{pmatrix}$. As $\Gamma(m)$ is an invariant subgroup of $\Gamma(1)$, there exists a substitution $M^{(m)}$ of $\Gamma(m)$ such that $\begin{pmatrix} P & Q \\ C_0P+V & C_0Q+W \end{pmatrix} M^{(m)}=\begin{pmatrix} UP_0 & UQ_0 \\ V_0 & W_0 \end{pmatrix}$.

Thus we get $(P, Q)M^{(m)}=(UP_0, UQ_0)$, where $M^{(m)}$ is a substitution of $\Gamma(m)$.

Let $\sigma(m)$ be the number of non-equivalent rational points by the group $\Gamma(m)$, that is the number of "rational vertices" of the fundamental domain of the group $\Gamma(m)$ in homogeneous coordinates.

We say that a modular form $f$ is in a rational point $(C, D)$ zero when the constant term in the Fourier expansion of $|CX+D|^{2r}f$ by the uniformization-variables, namely $\sum_I a(I)e^{\frac{2\pi i}{m}\sigma(I(AX+B)(CX+D)^{-1})}$, where $M=\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is a modular substitution, is zero and call a modular form of the *stufe* $m$ a vertex-form when it becomes zero in all rational vertices of the fundamental domain of the group $\Gamma(m)$.

The constant term of the Fourier expansion of the function

$$|CX+D|^{2r}f_r(X; K_1, L_1; m)$$
$$=f_r\left((AX+B)(CX+D)^{-1}; K_1D'-L_1C', -K_1B'+L_1A'; m\right)$$

in the rational point $(C, D)$ is not zero if and only if

$$(6) \qquad K_1D'-L_1C'\equiv 0\,, \qquad -K_1B'+L_1A'\equiv U \qquad \text{mod } m\,.$$

The condition (6′) is equivalent to

$$(6') \qquad K_1\equiv UC\,, \qquad L_1\equiv UD \qquad \text{mod } m\,.$$

It is independent of the choice of $A$, $B$.

There exists a trivial relation $f_r(X; UP_1, UQ_1; m)=f_r(X; P_1, Q_1; m)$, so that the function depends only on the rational point $(P_1, Q_1)$ and not on the individual symmetrical pair of matrices $P_1$, $Q_1$.

Therefore from (5), (5′) and (6′) we get the following

*Theorem* 1: There are exactly $\sigma(m)$ linearly independent Siegel's modular form of the degree $n$, of the dimension $-2r$, and of the *stufe* $m$ $f_r(X; P_i, Q_i; m)$ and a linear combination of $f_r(X; P_i, Q_i; m)$ of the same kind is identically zero if and only if it is a vertex-form.

*Theorem* 2: Any modular form of the degree $n$ of the dimension $-2r$, and of the *stufe* $m$ can be written uniquely as the sum of a linear combination of $f_r(X; P_i, Q_i; m)$ and a vertex-form of the same kind.