PAPERS COMMUNICATED

9. *On the Uniqueness of Haar's Measure.*

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1. For a topological group G, which is locally compact and separable, the uuiqueness[1] of Haar's left-invariant measure is proved by J. v. Neumann.[2] Although the method used by him is very interesting and powerful, his proof is somewhat long. The notion of right-zero-invariance is not necessary for the proof. In this paper we shall give a simplified proof. The essential improvement consists in the adoption of the *right*-invariant measure in the second group, in constructing the measure of the topological product $G \times G$. Since the separability plays no essential rôle in our proof, it can also be, by slight modifications, applied to the case of a non-separable group (the case of a locally bicompact topological group, which is treated by A. Weil[3]), and moreover we can prove, in the same manner, the theorem of the uniqueness of Haar's measure even for the case, when the field G is no longer a topological group, that is, G is simply a topological space S, and when the transitive group G of homeomorphic transformations of S on itself is given.[4]

2. We begin with some definitions:

Given a topological space S and a group G of homeomorphic transformation σ of S on itself, a set function[5] μ of S (not necessarily non-negative) is called *G-invariant* if for any Borel set E of S and for any $\sigma \in G$ we have $\mu(\sigma E) = \mu(E)$. A Borel set E of S is called *G-μ-invariant* if $\mu(E \Delta \sigma E) = 0$[6] for any $\sigma \in G$, and G is called *ergodic on S*, if for any G-invariant totally additive non-negative set function[7] μ and for any G-μ-invariant Borel set E of S we have either $\mu(E) = 0$ or $\mu(G - E) = 0$.[8] In the special case, when S and G coincide, that is, when S is a topo-

1) We shall understand in the following by "uniqueness" always "uniqueness up to a constant factor."

2) J. v. Neumann: The uniqueness of Haar's measure, Recueil Math., **1 (43)** (1936).

3) A. Weil: Sur les groupes topologiques et les groupes mesurés, C. R. **202** (1936).

4) Cf. J. v. Neumann: On the uniqueness of invariant Lebesgue measure, Bull. Amer. Math. Soc., **42** (1936) (Abstract).

5) In this paper we consider only those set functions, which are defined and are finite for all Borel sets whose closure is bicompact. (If the space is separable, the notion of compactness and that of bicompactness coincide.) If μ is non-negative, $\mu(E)$ is defined for any non-bicompact Borel set as the least upper bound of all $\mu(F)$, where F is a bicompact Borel set $\subset E$. $\mu(E)$ might be infinite in this case.

6) We denote by $A \Delta B$ the symmetric difference $A + B - A \cdot B$ of two sets A and B. This is the sum of two sets in Boolean sense.

7) We do not assume that $\mu(U) > 0$ for open set U.

8) Cf. J. v. Neumann and F. J. Murray: On rings of operators, Annals of Math., **37** (1936), p. 195.

logical group G and G is a group of left (right)-multiplication of G, G-invariant set function of S $(=G)$ is called *left (right)-invariant*, G-μ-invariant set is called *left (right)-μ-invariant* and G is called *left (right)-ergodic* if G is ergodic on itself, considering G as a group of transformations of itself by left (right)-multiplication.

With this notion of ergodicity, the proof of the theorem of uniqueness is now divided into two parts—the proof of the following two facts:

I. *If G is left-ergodic, then the left-invariant measure of G is unique,*

II. *G is left-ergodic,*

G being supposed to be locally compact and separable.[1)]

3. *Proof of I.* If the left-invariant measure of G is not unique, then there will be two left-invariant measures μ_1, μ_2, two Borel sets E_1, E_2 and a real number a such that $\mu_1(E_1) > a\mu_2(E_1)$ and $\mu_1(E_2) < a\mu_2(E_2)$. Consider the set function $\mu(E) = \mu_2(E) - a\mu_2(E)$. μ is defined for any Borel set and is left-invariant. Since μ is of bounded variation in any compact part of G, G can be divided into two Borel sets P and N: $G=P+N$, $P\cdot N=0$, in such a way that for any Borel set $E\subset P (E\subset N)$ we have $\mu(E)\geqq 0$ $\big(\mu(E)\leqq 0\big)$.[2)] Since μ is left-invariant, P and N are left-μ-invariant.

Now, consider the total variation $\bar{\mu}$ of μ. $\bar{\mu}$ is defined by $\bar{\mu}(E) = \mu(E\cdot P) - \mu(E\cdot N)$. Since $\bar{\mu}$ is totally additive, non-negative and left-invariant and since, as is easily seen, P and N are left-$\bar{\mu}$-invariant, we have by the assumption of ergodicity, that either $\bar{\mu}(P)=0$ or $\bar{\mu}(N)=0$. On the other hand, since there are two Borel sets E_1 and E_2 with $\mu(E_1)>0$ and $\mu(E_2)<0$, we have $\bar{\mu}(P)\geqq \mu(E_1\cdot P)\geqq \mu(E_1)>0$ and $\bar{\mu}(N)\geqq -\mu\ (E_2\cdot N)\geqq -\mu(E_2)>0$, and thus we are led to the contradiction.

Hence the left-invariant measure of G is unique.

4. *Proof of II.* Let μ be a left-invariant totally additive non-negative set function defined on G and let E be a left-μ-invariant Borel set of G. It is to be proved that either $\mu(E)=0$ or $\mu(G-E)=0$ holds.

Consider the product space $G\times G$ and introduce into it a measure $\mu+\lambda$, which is induced by the measure μ of the x-axis (the first G) and by the measure λ of the a-axis (the second G). As the measure μ of the x-axis take the left-invariant measure μ in question and as

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1) Since the case of a locally bicompact topological group is treated in § 5, the treatment of a locally compact separable group is superfluous. But the relation between uniqueness and ergodicity is more prominent in this case.

2) Here we use the conditions of separability and of local compactness. If G is compact, then μ is of bounded variation on G and the proof is immediate. Let K be the least upper bound of $\mu(E)$ for all Borel sets E of G, and choose a sequence $\{E_n\}$ of Borel sets of G such that $\mu(E_n)>K-1/2^n$, $(n=1, 2, \ldots\ldots)$. It will be clear, that $G=P+N$, where $P=\overline{\lim\limits_{n\to\infty}} E_n$ and $N=G-P$, gives the desired decomposition. If G is not compact, we proceed as follows: Since G is separable and locally compact, G is divided into a countable number of (not necessarily disjoint) open sets U_i $(i=1, 2, \ldots\ldots)$ whose closure is compact. Each U_i is then divided into positive part P_i and negative part N_i: $U_i=P_i+N_i$, $P_i\cdot N_i=0$, and if we put $P=\sum_{i=1}^{\infty} P_i$ and $N=G-P$, $G=P+N$ will be the desired decomposition.

the measure λ of the a-axis take an arbitrary right-invariant measure of G.

Let $\varphi(x)$ be the characteristic function of E (defined in G). $\varphi(x)$ is a Borel-measurable function in G and consequently $\varphi(ax)$ is Borel-measurable in $G\times G$. $\varphi(x)$ itself, considered as a function of x and a, is also Borel-measurable in $G\times G$. Since, as is easily seen, $|\varphi(x)-\varphi(ax)|$ (a: fixed) is a characteristic function of $E\varDelta a^{-1}E$ and since, by assumption $\mu(E\varDelta aE)=0$ for any $a\in G$, we have

$$\int_G |\varphi(x)-\varphi(ax)|\,d\mu(x)=0$$

for any $a\in G$, and consequently

$$\iint_{G\times G} |\varphi(x)-\varphi(ax)|\,d\mu(x)d\lambda(a)=0\,.$$

Therefore, applying the theorem of Fubini:

$$\int_G |\varphi(x)-\varphi(ax)|\,d\lambda(a)=0$$

for almost all x (with respect to μ) of G. In other words, there is a Borel set M_0 of μ-measure zero on x-axis, such that for any $x\in G-M_0$ there corresponds a Borel set E_x of λ-measure zero on the a-axis, for which from $a\in G-E_x$ it follows $\varphi(x)=\varphi(ax)$, or equivalently $\varphi(x)=\varphi(z)$ for any $z\in G-E_x\cdot x$. Let y be any other point of $G-M_0$. Then there corresponds, in the same manner, a set E_y of λ-measure zero on the a-axis, for which from $z\in G-E_y\cdot y$ it follows $\varphi(y)=\varphi(z)$. Since the set $E_x\cdot x$ and $E_y\cdot y$ of excluded points, by the right-invariance of λ, is of λ-measure zero, there will be a point $z\in G-E_x\cdot x-E_y\cdot y$ and for this z we have $\varphi(x)=\varphi(y)=\varphi(z)$. Since x and y are arbitrarily chosen from $G-M_0$, we have thus proved that $\varphi(x)$ is constant in $G-M_0$, that is, either $E\subset M_0$ or $G-M_0\subset E$ and consequently either $\mu(E)=0$ or $\mu(G-E)=0$.

The uniqueness of the left-invariant measure is hereby completely proved.

5. The case of a locally bicompact topological group is now to be treated. It is reported by A. Weil[1] that he has succeeded in proving the existence and the uniqueness of Haar's measure for such a group. Since the detail of his proof is not published anywhere, we shall give in this section a brief summary of our proof.

The existence of Haar's measure in any locally bicompact topological group can be proved in just the same manner, as it was done by S. Banach[2] for the case of a locally compact separable group. To our regret, we have to rely upon the well-ordering hypothesis. Since the

1) A. Weil, loc. cit. As is remarked by J. v. Neumann, the method used by A. Weil is entirely different from that of v. Neumann. (Cf. J. v. Neumann, loc. cit., p. 723).

2) Cf. Appendix of S. Saks' Book: The Theory of the Integral, Warszawa-Lwow, 1937.

necessary modifications are almost obvious, we shall not go into the detail and shall proceed to the proof of the theorem of uniqueness.

If we follow the preceding proof, we are hindered at a point, where we divide G into positive part P and negative part N with respect to μ. Indeed, since G is in general not semi-bicompact (that is, G is not a sum of a countable number of bicompact sets), the decomposition $G=\sum U_a$ of G into open sets U_a, whose closure is bicompact, might perhaps require more than a countable number of such sets; in such a case, $P=\sum P_a$, where P_a is a positive part of U_a, might no longer be Borel-measurable and moreover there might exist a bicompact Borel set $E \subset P$ such that $\mu(E) < 0$.

In order to avoid such complications, we shall consider only a local part of G. Take an arbitrary open set U whose closure is bicompact, such that $U^{-1}=U$. It will be sufficient, if we can show that the assumption of the existence of a totally additive left-invariant set function μ and two Borel sets E_1, $E_2 \subset U$, such that $\mu(E_1) > 0$ and $\mu(E_2) < 0$, would lead us to the contradiction.

Consider the open set U^3.[1] $\overline{U^3}$ is bicompact and can therefore be divided into positive part P and negative part N with respect to μ. Since μ is left-invariant we have $\mu\big((P\Delta aP)\cdot U\big)=0$ for any $a \in U^2$.[2] Consider the total variation $\bar{\mu}$ of μ again. We have also $\bar{\mu}\big((P\Delta aP)\cdot U\big)=0$ for any $a \in U^2$ and, denoting by $\varphi(x)$ the characteristic function of P, we have

$$\iint\limits_{U\times U^2} |\varphi(x)-\varphi(ax)|\, d\mu(x) d\lambda(a)$$

for any right invariant measure λ of G. Therefore we have, from the theorem of Fubini, the existence of a Borel set $M_0 \subset U$ with $\mu(M_0)=0$, such that

$$\int\limits_{U^2} |\varphi(x)-\varphi(ax)|\, d\lambda(a)=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

for any $x \in U-M_0$; that is, there corresponds to any $x \in U-M_0$ a Borel set E_x of λ-measure zero, such that from $a \in U^2-E_x$ it follows $\varphi(x)=\varphi(ax)$, or equivalently, $\varphi(x)=\varphi(z)$ for any $z \in (U^2-E_x)\cdot x$. Since $U \subset U^2\cdot x$ and $\lambda(E_x\cdot x)=0$ (owing to the right-invariance of λ), we have $\varphi(x)=\varphi(z)$ for any $z \in U-E_x^*$ with $\lambda(E_x^*)=0$.

The rest of the proof will now be obvious, since, using the condition that $\lambda(U) > 0$, it can be performed in just the same manner as in the preceding.

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1) U^n is a set of all x of G, which is representable as $x=x_1\cdot x_2\ldots\ldots x_n$, where $x_i \in U$ $(i=1, 2, \ldots\ldots, n)$.

2) This relation may be proved as follows: Since $(P-aP)\cdot U \subset P$ we have $\mu((P-aP)\cdot U) \geqq 0$ and since $(P-aP)\cdot U \subset U-aP \subset aU^3-aP=aN$, we have $\mu((P-aP)\cdot U) \leqq 0$, and combining these two, we have $\mu((P-aP)\cdot U)=0$. Since the relation $\mu((aP-P)\cdot U)=0$ may be proved in the same manner, we have $\mu((P\Delta aP)\cdot U)$ $=\mu((P-aP)\cdot U)+\mu((aP-P)\cdot U)=0$.

Remark.[1] The last part of the proof may also be done as follows:[2] From (1) we have

$$\int_{U^2} \varphi(x)\,|\,\varphi(x)-\varphi(ax)\,|\,d\lambda(a)=\varphi(x)\int_{U^2}\big(1-\varphi(ax)\big)d\lambda(a)=0\,.$$

Since λ is right-invariant

$$\varphi(x)\int_{U^2x}\big(1-\varphi(a)\big)d\lambda(a)=0\,,$$

and since $U\subset U^2\cdot x$ for any $x\in U$

$$\varphi(x)\int_{U}\big(1-\varphi(a)\big)d\lambda(a)=0\,. \qquad \text{..................} \qquad (2)$$

In the same manner, we have from (1), multiplying the integrand by $\varphi(ax)$,

$$\big(1-\varphi(x)\big)\int_{U}\varphi(a)d\lambda(a)=0\,. \qquad \text{..................} \qquad (3)$$

Since (2) and (3) hold for any $x\in U-M_0$, and from the relation $\int_U\big(1-\varphi(a)\big)d\lambda(a)+\int_U\varphi(a)d\lambda(a)=\lambda(U)>0$, at least one of $\int_U\big(1-\varphi(a)\big)d\lambda(a)$ and $\int_U\varphi(a)d\lambda(a)$ must be positive, we have from (2) or (3) $\varphi(x)\equiv1$ or 0 in $U-M_0$.

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1) The author wishes to express his hearty thanks to Mr. M. Fukamiya for his kind advices concerning this method.

2) This method is also applicable to the case of a locally compact separable group.