75. *Iteration of Linear Operations in Complex Banach Spaces.*

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§ **1.** Introduction. In a recent paper C. Visser[1] discussed the iteration of linear operations in a Hilbert space and proved the following theorems:

Theorem I. Let E be a Hilbert space and A a linear operator which maps E in itself. If $\{A^n\}$ $(n=1, 2, \dots)$ *is uniformly bounded, then there exists a bounded linear operator* A_1, *which maps E in itself, such that* $\frac{1}{n}(A+A^2+\cdots+A^n)$ *converges weakly to* A_1.

Theorem II. If in addition to the assumptions in Theorem I, A is completely continuous, then the weak convergence in Theorem I may be substituted by the strong convergence.

In Visser's proof, the notion of inner product is indispensable, and therefore it is not applicable to the case of general Banach spaces. It is the purpose of the present paper to show that these theorems are also valid in general complex Banach spaces. Moreover, we can show that even in general complex Banach spaces the *weak* convergence in Theorem I may be substituted by the *strong* one (Theorem 2)[2] and that the *strong* convergence in Theorem II by the *uniform* one (Theorem 4). The former is a generalisation of J. v. Neumann's Mean Ergodic Theorem[3] and the latter is an analogue of M. Fréchet's theorem;[4] and following the same way, the results of N. Kryloff and N. Bogolioùboff[5] are also generalised to the case of general complex Banach spaces (Theorem 5).

The results of this paper are obtained in collaboration with K. Yosida. Theorem 2 is obtained by him directly,[6] and Theorem 4 and 5 are also obtained by him in different ways.[7] I shall, however, give the outline of my proof, since we can treat all these problems in a unique way and since the method itself, I believe, is not without interest. In concluding the introduction, I should like to express my hearty thanks to K. Yosida for his kindness in the course of this work.

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1) C. Visser: On the iteration of linear operations in a Hilbert space, Proc. Acad. Amsterdam, **41** (1938), 487–495.

2) This is an essential advance! This is proved by K. Yosida. See the foregoing paper of K. Yosida: Mean Ergodic Theorem in Banach spaces.

3) J. v. Neumann: Proof of the quasi-ergodic hypothesis, Proc. Nat. Acad. U.S.A. **18** (1932), 70–82.

4) M. Fréchet: Sur l'allure asymptotique de la suite des itérés d'un noyau de Fredholm, Quart. Journ. of Math., **5** (1934), 106–144.

5) N. Kryloff et N. Bogolioùboff: Sur les probabilités en chaîne, C. R., **204** (1937), 1386–1388.

6) See the paper of K. Yosida cited in (2).

7) See the foregoing paper of K. Yosida: Abstract integral equation and the homogeneous stochastic process.

§ **2.** Let E be a complex Banach space and A a bounded linear operator which maps E in itself. A is called to be *weakly completely continuous* if it maps the unit sphere $\|x\| \leqq 1$ of E on a weakly compact (in E) set of E.

Theorem 1. Let A be a weakly completely continuous operator which maps E in itself. If there is a constant C such that $\|A^n\| \leqq C$ for $n=1, 2, \ldots$, then there exists a bounded linear operator A_1, which maps E in itself, such that

(i) $\frac{1}{n}(A+A^2+\cdots+A^n)$ *converges weakly to* A_1,

(ii) $\|A_1\| \leqq C$,

(iii) $A^n A_1 = A_1 A^n = A_1$ *for* $n=1, 2, \ldots$,

(iv) $A_1^2 = A_1$.

Proof. We may assume without the loss of generality that E is separable, since otherwise we have only to consider, for any $x \in E$, the closed linear subspace E_x of E which is spanned by $\{A^n x\}$ $(n=1, 2, \ldots)$.[1] Since $\left\{\frac{1}{n}(A+A^2+\cdots+A^n)x\right\}$ $(n=1, 2, \ldots)$ is weakly compact for any $x \in E$, we can choose (diagonal method!) a subsequence $\{n_\nu\}$ $(\nu=1, 2, \ldots)$ of $\{n\}$ $(n=1, 2, \ldots)$ such that $\frac{1}{n_\nu}(A+A^2+\cdots+A^{n_\nu})x$ $(\nu=1, 2, \ldots)$ converges weakly to a point of E at a countable subset which is dense in E. Since $\left\{\frac{1}{n_\nu}(A+A^2+\cdots+A^{n_\nu})\right\}$ is uniformly bounded, we can see from this that $\frac{1}{n_\nu}(A+A^2+\cdots+A^{n_\nu})x$ converges weakly to a point (say $A_1 x$) of E for any point x of E; that is, $\frac{1}{n_\nu}(A+A^2+\cdots+A^{n_\nu})$ converges weakly to a linear operator A_1 which maps E in itself.

Thus the linear operator A_1 is determined. We have, however, hitherto proved only that

(i)′ $\frac{1}{n_\nu}(A+A^2+\cdots+A^{n_\nu})$ converges weakly to A_1,

and yet (i) is not proved. Before coming to the proof of (i), let us consider the properties (ii), (iii) and (iv). (ii) is evident. In order to prove (iii), multiply (i)′ by A from the left. Then we have:

$$\frac{1}{n_\nu}(A^2+A^3+\cdots+A^{n_\nu+1}) \text{ converges weakly to } AA_1.$$

Consequently, since

$$\left\|\frac{1}{n_\nu}(A+A^2+\cdots+A^{n_\nu})-\frac{1}{n_\nu}(A^2+A^3+\cdots+A^{n_\nu+1})\right\|$$
$$=\left\|\frac{1}{n_\nu}(A-A^{n_\nu+1})\right\| \leqq \frac{2C}{n_\nu} \to 0,$$

1) It will be easily seen that A is a weakly completely continuous operator which maps E_x in itself. (By a theorem of Banach-Mazur, E_x is also weakly closed!).

we have (with (i)$'$) that $AA_1=A_1$. The relation $A_1A=A_1$ may be proved analogously. Hence (iii) is proved for $n=1$, and the case for general n follows from this directly. If we multiply (i)$'$ again by A_1 from the left, then we have:

$$\frac{1}{n_\nu}(A_1A+A_1A^2+\cdots+A_1A^{n_\nu}) \text{ converges weakly to } A_1^2.$$

Since by (iii) $A_1A^n=A_1$ for $n=1,2,\ldots$, the left hand side is equal to A_1 and hence (iv) is proved.

Now, in order to prove (i), assume that (i) is not true. Then there is a point $x_0 \in E$ and a subsequence $\{m_\nu\}$ $(\nu=1,2,\ldots)$ of $\{n\}$ $(n=1,2,\ldots)$ such that $\frac{1}{m_\nu}(A+A^2+\cdots+A^{m_\nu})x_0$ converges weakly to $y_0 \neq A_1x_0$. Starting from this sequence $\{m_\nu\}$ $(\nu=1,2,\ldots)$ and applying the diagonal method again, we shall have a subsequence $\{n'_\nu\}(\nu=1,2,\ldots)$ of $\{m_\nu\}$ $(\nu=1,2,\ldots)$ and a bounded linear operator A'_1, which maps E in itself, such that

$$\text{(i)}'' \quad \frac{1}{n'_\nu}(A+A^2+\cdots+A^{n'_\nu}) \text{ converges weakly to } A'_1.$$

A'_1 satisfies the same properties (ii), (iii) and (iv) as A_1, and $A_1 \neq A'_1$ since $A'_1x_0=y_0 \neq A_1x_0$. If we now multiply (i)$''$ by A_1 from the left, then we have (using $A_1A^n=A_1$) that $A_1=A_1A'_1$. On the other hand, multiplying (i)$'$ by A'_1 from the right, we have $A'_1=A_1A'_1$, which leads to a contradiction since $A_1 \neq A'_1$. Hence (i) must be true.

Theorem 2. The weak convergence in Theorem 1 may be substituted by the strong convergence.

Proof. Let the range of $I-A$ (I is an identical transformation) and of $I-A_1$ be R and R_1 respectively. It is clear that any point of R_1 is a weak limiting point of R,[1] and since R is a linear subspace, it is also a strong limiting point of R.[2] Hence we have $R_1 \subset \bar{R}$.

Now, put $x=A_1x+(x-A_1x)$ for any $x \in E$. Since $A^nA_1x=A_1x$ for $n=1,2,\ldots$ and since $x-A_1x \in R_1 \subset \bar{R}$, the proof will be completed if we can prove that $\left\|\frac{1}{n}(A+A^2+\cdots+A^n)(x-A_1x)\right\| \to 0$ or, more generally, that for any $y \in \bar{R}$ we have $\left\|\frac{1}{n}(A+A^2+\cdots+A^n)y\right\| \to 0$. This is clear if $y \in R$; for, since $y=x-Ax$ with $x \in E$, we have $\left\|\frac{1}{n}(A+A^2+\cdots+A^n)y\right\| = \left\|\frac{1}{n}(A+A^2+\cdots+A^n)(x-Ax)\right\| = \left\|\frac{1}{n}(A-A^{n+1})x\right\| \to 0$. If in general $y \in \bar{R}$, then for any $\varepsilon > 0$ there exists a $y^* \in R$ with $\|y-y^*\| < \varepsilon$. For this y^* we have:

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1) Consider that $I-\frac{1}{n}(A+A^2+\cdots+A^n)=\frac{1}{n}(I-A)(nI+(n-1)A+(n-2)A^2+\cdots+A^{n-1})$ converges weakly to $I-A_1$.

2) Theorem of Banach-Mazur.

$$\overline{\lim_{n\to\infty}}\left\|\frac{1}{n}(A+A^2+\cdots+A^n)y\right\|\leqq\overline{\lim_{n\to\infty}}\left\|\frac{1}{n}(A+A^2+\cdots+A^n)y^*\right\|$$
$$+\overline{\lim_{n\to\infty}}\left\|\frac{1}{n}(A+A^2+\cdots+A^n)(y-y^*)\right\|\leqq C\cdot\varepsilon$$

and since $\varepsilon>0$ is arbitrary, we have $\left\|\frac{1}{n}(A+A^2+\cdots+A^n)y\right\|\to 0$.

Theorem 3. Under the same assumptions as in Theorem 1 we have: For any complex number λ with $|\lambda|=1$ there exists a bounded linear operator A_λ, which maps E in itself, such that

(i) $\frac{1}{n}\left(\frac{A}{\lambda}+\frac{A^2}{\lambda^2}+\cdots+\frac{A^n}{\lambda^n}\right)$ *converges strongly to* A_λ,

(ii) $\|A_\lambda\|\leqq C$,

(iii) $A^nA_\lambda=A_\lambda A^n=\lambda^n A_\lambda$ *for* $n=1, 2, \ldots$,

(iv) $A_\lambda^2=A_\lambda$,

(v) $\lambda\neq\mu$ *implies* $A_\lambda A_\mu=A_\mu A_\lambda=0$,

(vi) $A_\lambda\neq 0$ *if and only if* λ *is a proper value of* A.

If we further put $A'=A-\sum_{i=1}^{p}\lambda_i A_{\lambda_i}$ *for any* λ_i *with* $\lambda_i\neq\lambda_j$ $(i\neq j)$ *and* $A_{\lambda_i}\neq 0$ $(i=1, 2, \ldots, p)$, *then we have*

(vii) $A_{\lambda_i}A'=A'A_{\lambda_i}=0$ *for* $i=1, 2, \ldots, p$,

(viii) $AA'=A'A=A'^2$,

(ix) $A^n=\sum_{i=1}^{p}\lambda_i^n A_{\lambda_i}+A'^n$ *for* $n=1, 2, \ldots$, *and there exists a constant* C' *such that we have* $\|A'^n\|\leqq C'$ *for* $n=1, 2, \ldots$,

(x) λ *is a proper value of* A' *if and only if it is a proper value of* A *and* $\lambda\neq\lambda_i$ *for* $i=1, 2, \ldots, p$.

We omit the proof.

§ 3. In this chapter we are concerned with completely continuous operators. A linear transformation of a complex Banach space E in itself is called to be *completely continuous* if it maps the unit sphere $\|x\|\leqq 1$ of E on a compact (in E) set of E.

Theorem 4. Let A be a completely continuous linear operator which maps E in itself. If there is a constant C such that $\|A^n\|\leqq C$ for $n=1, 2, \ldots$, then we have:

(i) *For any complex number λ with $|\lambda|=1$ there exists a completely continuous linear operator A_λ, which maps E in itself, such that*

$$\left\|\frac{1}{n}\left(\frac{A}{\lambda}+\frac{A^2}{\lambda^2}+\cdots+\frac{A^n}{\lambda^n}\right)-A_\lambda\right\|\leqq\frac{M}{n}\quad \textit{for}\quad n=1, 2, \ldots,$$

where M is a constant which is independent of n. Moreover, A_λ has the properties (ii)–(x) *of Theorem 3.*

(ii) *In order that A^n converges uniformly to a zero operator, it is necessary and sufficient that A has no proper values of absolute value* 1.

(iii) *In order that A^n converges uniformly to a linear operator*

$A_1 \neq 0$, *it is necessary and sufficient that* A *has a proper value* 1 *and no proper values of absolute value* 1.

In (ii) *and* (iii) *if* A^n *converges uniformly, it is of the order of geometrical progression; that is, there are a positive number* δ *and a constant* M *independent of* n *such that we have* $\|A^n\| \leqq M \cdot (1-\delta)^n$ *and* $\|A^n - A_1\| \leqq M \cdot (1-\delta)^n$ *respectively for* $n=1, 2, \ldots$.

Proof. By a theorem of F. Riesz[1] the proper values of a completely continuous linear operator do not accumulate to a point $\lambda \neq 0$. Hence the proper values λ of A with $|\lambda|=1$ are finite in number, and if we denote these by $\lambda_1, \lambda_2, \ldots, \lambda_p$ and consider the linear operator $A' = A - \sum_{i=1}^{p} \lambda_i A_{\lambda_i}$, then there is a constant $\delta > 0$ such that A' has no proper values λ with $|\lambda| \geqq 1-\delta$.[2] Consequently there is a constant M such that $\|A'^n\| \leqq M \cdot (1-\delta)^n$ for $n=1, 2, \ldots$. The rest of the proof will now be almost obvious, if we consider Theorem 3 (especially property (ix)).[3]

We shall now proceed to the generalisation of a theorem of Kryloff and Bogolioùboff.

Theorem 5. Let A *be a bounded linear operator which maps* E *in itself. If there is a constant* C *such that* $\|A^n\| \leqq C$ *for* $n=1, 2, \ldots$ *and if there are an integer* k *and a completely continuous linear operator* V *such that* $\|A^k - V\| = \alpha < 1$, *then for any complex number* λ *with* $|\lambda|=1$ *there exists a completely continuous linear operator* A_λ, *which maps* E *in itself, such that*

$$\left\| \frac{1}{n}\left(\frac{A}{\lambda} + \frac{A^2}{\lambda^2} + \cdots + \frac{A^n}{\lambda^n}\right) - A_\lambda \right\| \leqq \frac{M}{n} \quad \textit{for} \quad n=1, 2, \ldots,$$

where M *is a constant which is independent of* n.

Moreover, all what we have obtained in Theorem 3 and 4 is also true for this case, except the fact that A' *is completely continuous.*

Proof. It will be sufficient if we prove the case $k=1$. We shall prove first that $\frac{1}{n}(A + A^2 + \cdots + A^n)$ converges strongly to a linear operator A_1. For this purpose it will be sufficient if we can prove that for any $x \in E$ the set $\left\{\frac{1}{n}(A + A^2 + \cdots + A^n)x\right\}$ $(n=1, 2, \ldots)$ is totally bounded; for all what we have needed is the existence of a (weakly or strongly) convergent subsequence. In order to prove the total boundedness, let $\varepsilon > 0$ be an arbitrary positive number and consider the linear operator $V_p = A^p - (A-V)^p$, where p is an integer such that $C \cdot \alpha^p \cdot \|x\| < \frac{\varepsilon}{3}$. V_p is completely continuous, since expanding the right hand side the term A^p vanishes and there remain only those terms which contain at least one V-factor. Take an n_0 so large that

1) F. Riesz: Über lineare Funktionalgleichungen, Acta Math., **41** (1918), 71–98.

2) Since $\{A^n\}$ is uniformly bounded, A has no proper values λ with $|\lambda| > 1$.

3) It will be easily seen that A_λ and A' are completely continuous.

$\frac{p}{n_0}\cdot C\cdot\|x\|<\frac{\varepsilon}{3}$, then we have for $n>\max(n_0, p)$

$$(*)\quad \left\|\frac{1}{n}(A+A^2+\cdots+A^n)x-V_p\Big(\frac{1}{n}(A+A^2+\cdots+A^{n-p})x\Big)\right\|$$

$$\leqq\left\|\frac{1}{n}(A+A^2+\cdots+A^p)x\right\|+\left\|(A-V)^p\Big(\frac{1}{n}(A+A^2+\cdots+A^{n-p})x\Big)\right\|$$

$$\leqq\frac{p}{n_0}\cdot C\cdot\|x\|+\alpha^p\cdot\frac{n-p}{n}\cdot C\cdot\|x\|<\frac{2\varepsilon}{3},$$

and, since the set $\Big\{V_p\Big(\frac{1}{n}(A+A^2+\cdots+A^{n-p})x\Big)\Big\}$ $\big(n>\max(n_0, p)\big)$ is totally bounded (and consequently $\frac{\varepsilon}{3}$-bounded), so is the set $\Big\{\frac{1}{n}(A+A^2+\cdots+A^n)x\Big\}$ $\big(n>\max(n_0, p)\big)$ ε-bounded. Since the excluded points are finite in number and since $\varepsilon>0$ is arbitrary, the set $\Big\{\frac{1}{n}(A+A^2+\cdots+A^n)x\Big\}$ $(n=1, 2, \ldots)$ is totally bounded.

Thus we have proved the existence of A_1 and the strong convergence to it. This is true also for A_λ with $|\lambda|=1$. It will be easily seen from (*) that A_λ is completely continuous. A_λ has clearly the properties (ii)–(x) of Theorem 3.

In order to substitute the strong convergence by the uniform one, let us proceed as follows: By a theorem of K. Yosida,[1] the proper values of A do not accumulate to a point λ with $|\lambda|=1$. Hence the proper values λ of A with $|\lambda|=1$ are finite in number, and if we denote these by $\lambda_1, \lambda_2, \ldots, \lambda_p$ and consider the linear operator $A'=A-\sum_{i=1}^{p}\lambda_i A_{\lambda_i}$, then there is a constant $\delta>0$ such that A' has no proper values λ with $|\lambda|\geqq 1-\delta$. Since there is a completely continuous operator $V'=V-\sum_{i=1}^{p}\lambda_i A_{\lambda_i}$ such that $\|A'-V'\|=\|A-V\|=\alpha<1$, we have again from a theorem of K. Yosida,[2] that $E-\frac{1}{\lambda}A'$ has an inverse for $|\lambda|\geqq\max\Big(1-\delta, \frac{1}{2\alpha}\Big)\equiv 1-\delta'$, and therefore the series $\sum_{n=0}^{\infty}\frac{A'^n}{\lambda^n}$ converges for $|\lambda|\geqq 1-\delta'$. Hence there exists a constant M such that $\|A'^n\|\leqq M\cdot(1-\delta')^n$ for $n=1, 2, \ldots$.

The rest of the proof will be easily carried out as in the preceding.

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1) Lemma 1, loc. cit. (7).

2) Lemma 3, loc. cit. (7). This may also be proved directly.