# 25. *On the Strong Law of Large Numbers.*

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**1.** Let

$$X_1, X_2, \dots, X_n, \dots \tag{1}$$

be a sequence of independent chance variables and let the expectation of $X_n$, $E(X_n)$ be 0. If

$$\frac{s_n}{n} = \frac{X_1+X_2+\cdots+X_n}{n} \tag{2}$$

converges to zero with probability 1, we say that the sequence (1) obeys to the strong law of large numbers. The ordinary law of large numbers asserts that (2) converges in probability.

Sufficient conditions for the validity of the strong law of large numbers were given by various authors.[1]

Concerning the series of independent chance variables, it is well known that the convergence in probability and the convergence with probability 1 is equivalent. This is due to P. Lévy. Mr. G. Ottaviani has recently given a simple proof[3] of this theorem. For the sequence (2), the similar facts do not necessarily hold. Thus it arises the problem to give the condition to conclude the validity of the strong law from that of the ordinary law of large numbers. The present paper concerns this problem.

**2.** *Theorem 1. For any positive* $\varepsilon$, *let*

$$Pr\left(\varepsilon > \frac{s_n}{n} > -\varepsilon\right) > 1 - \delta_n(\varepsilon), \qquad \delta_n(\varepsilon) \to o \quad as \quad n \to \infty,$$

*and suppose that for any* $\varepsilon > o$

$$\sum_{k=1}^{\infty} \delta_{2^k}(\varepsilon) < \infty .$$

*Then the sequence* (1) *converges to zero with probability* 1.

To prove the theorem we use the method of G. Ottaviani. We shall prove the theorem in terms of Lebesgue measure in place of probability and let $X_i = X_i(t)$ $(i=1, 2, \dots)$ are measurable functions independent mutually.

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1) A. Kolmogoroff, Sur la loi forte des grands nombres, Comptes Rendus, **191** (1930), pp. 910–911.

J. Marcinkiewicz—A. Zygmund, Sur les fonctions indépendentes, Fund, Math., **29** (1937).

P. R. Halmos, On a necessary condition for the strong law of large numbers, Ann. Math., **40** (1939).

2) P. Lévy, Théorie de l'addition des variables aléatories (1937), p. 139.

3) G. Ottaviani, Giornali di Mat. (1939). 吉田耕作, 全國紙上數學談話會誌, **188**.

Let $2n \geqq m \geqq n$. If

$$s_m(t) > \varepsilon m, \qquad \sum_{m+1}^{2n} X_i(t) > -\frac{\varepsilon n}{2},$$

then clearly

$$s_{2n}(t) = \sum_{m+1}^{2n} X_i(t) + s_m(t) > \varepsilon\left(m - \frac{n}{2}\right) \geqq \frac{\varepsilon n}{2}.$$

Thus

$$(3) \qquad \sum_{m=n}^{2n} \prod_{i=n}^{m-1} E\left(s_i(t) \leqq \varepsilon i\right) \cdot E\left(s_m(t) > \varepsilon m\right) \cdot E\left(\sum_{i=m+1}^{2n} X_i(t) > -\frac{\varepsilon n}{2}\right)$$

$$\subset E\left(s_{2n}(t) > \frac{\varepsilon n}{2}\right).^{1)}$$

In $\prod_{i=n}^{m-1}$, the case $m=n$ means the set such that $s_n(t) > \varepsilon n$. If $\varepsilon n/4 > s_{2n}(t) > -\varepsilon n/4$, $\varepsilon m/8 > s_m(t) > -\varepsilon m/8$, then $s_{2n}(t) - s_m(t) > -\varepsilon n/4 - \varepsilon m/8 \geqq -\varepsilon n/2$. Thus we have

$$E\left(\sum_{i=m+1}^{2n} X_i(t) > -\frac{\varepsilon n}{2}\right)$$

$$\supset E\left(\frac{\varepsilon n}{4} > s_{2n}(t) > -\frac{\varepsilon n}{4}\right) \cdot E\left(\frac{\varepsilon m}{8} > s_m(t) > -\frac{\varepsilon m}{8}\right).$$

The measure of the last set exceed

$$1 - \delta_{2n}\left(\frac{\varepsilon}{8}\right) - \delta_m\left(\frac{\varepsilon}{8}\right) \geqq 1 - 2 \max_{n \leqq r \leqq 2n} \delta_r\left(\frac{\varepsilon}{8}\right)$$

$$\equiv 1 - \eta_n \qquad \left(\eta_n = \eta_n\left(\frac{\varepsilon}{8}\right)\right).$$

By independence, we have

$$(1 - 2\eta_n) \left| \sum_{m=n}^{2n} \prod_{i=n}^{m-1} E\left(s_i(t) \leqq \varepsilon i\right) \cdot E\left(s_m(t) > \varepsilon m\right) \right|$$

$$\leqq \left| E\left(s_{2n}(t) > \frac{\varepsilon n}{2}\right) \right|.^{2)}$$

The set in the bracket of the left hand side is the set such that there holds $s_i(t) > \varepsilon i$ for at least one $i (n \leqq i \leqq 2n)$.

Thus if we denote this set $e_n(\varepsilon)$, then

$$(1 - 2\eta_n) \mid e_n(\varepsilon) \mid \leqq \left| E\left(s_{2n}(t) > \frac{\varepsilon n}{2}\right) \right|.$$

Similarly we have

$$(1 - 2\eta_n) \mid e_n'(\varepsilon) \mid \leqq \left| E\left(s_{2n}(t) < -\frac{\varepsilon n}{2}\right) \right|,$$

where $e_n'(\varepsilon)$ is the set such that for at least one $i (n \leqq i \leqq 2n) s_i(t) < -\varepsilon i$. If $E_n(\varepsilon)$ denotes the set such that for every $i (n \leqq i \leqq 2n) \mid s_i(t) \mid \leqq \varepsilon i$ holds, then

1) $E(P(t))$ means the set of $t$ such that $P(t)$ holds.

2) $\mid E \mid$ means the measure of $E$.

$$
\begin{aligned}
|E_n(\varepsilon)| &= 1-|e_n(\varepsilon)+e'_n(\varepsilon)| \\
&= 1-|e_n(\varepsilon)|-|e'_n(\varepsilon)| \\
&\geqq 1-\frac{1}{1-2\eta_n}\left\{\left|E\left(s_{2n}(t)>\frac{\varepsilon n}{2}\right)\right|+\left|E\left(s_{2n}(t)<-\frac{\varepsilon n}{2}\right)\right|\right\} \\
&\geqq 1-\frac{\delta_{2n}\left(\frac{\varepsilon}{4}\right)}{1-2\eta_n} \\
&\leqq 1-\theta\delta_{2n}\left(\frac{\varepsilon}{4}\right)
\end{aligned}
$$

for $\eta_n \to 0$ as $n\to\infty$, where $\theta$ is some constant independent of $n$ for $n\geqq n_0$. In other words, the measure of the set such that $|s_i(t)|>\varepsilon i$ holds for every $i(n\leqq i\leqq 2n,\ n\geqq n_0)$ does not exceed $\theta\delta_{2n}\left(\frac{\varepsilon}{4}\right)$.

Take $k_0$ such that $2^{k_0}\geqq n_0$ and take $2^{k_0}, 2^{k_0+1}, \dots$ as the values of $n$ successively. Then we obtain that the measure of the set such that $|s_i(t)|>\varepsilon i$ holds for every $i(i\geqq 2^{k_0})$ does not exceed $\theta\sum_{i=k_0}^{\infty}\delta_{2^k}\left(\frac{\varepsilon}{4}\right)$ which is arbitrarilly small by taking $k_0$ large. By the usual way the conclusion of the theorem follows.

**3.** We now prove the following theorem due to A. Kolmogoroff[1) by use of Theorem 1.

*Theorem 2. Let* $E(X_n^2)=b_n$. *If*

$$
\sum_{n=1}^{\infty}\frac{b_n}{n^2}<\infty,
$$

*then the sequence* (1) *obeys to the strong law of large numbers.*

The theorem of this kind can be easily generalized to the case where $E(X_n^2)$ is not finite by the method of equivalence due to Khintchine and Kolmogoroff. But we do not concern it here.[2)

The proof is done as follows.

$E(s_n^2)=\sum_{i=1}^{n}b_i$ and Tchebycheff inequality show that

$$
(4)\qquad P_r(|s_n|\geqq nR)\leqq\frac{1}{n^2R^2}\sum_{i=1}^{n}b_i,\qquad (R>0).
$$

Hence

$$
\begin{aligned}
\sum_{1}^{\infty}\delta_{2^k} &\leqq \frac{1}{\varepsilon^2}\sum_{k=1}^{\infty}\frac{1}{2^{2k}}\sum_{i=1}^{2^k}b_i \\
&\leqq \frac{1}{\varepsilon^2}\sum_{k=1}^{\infty}\sum_{m=k}^{\infty}\left(\frac{1}{2^m}\right)^2\cdot\sum_{i=2^k+1}^{2^{k+1}}b_i \\
&\leqq \frac{1}{\varepsilon^2}\sum_{k=1}^{\infty}\left(\frac{1}{2^{k+1}}\right)^2\sum_{i=2^k+1}^{2^{k+1}}b_i\leqq\frac{8}{\varepsilon^2}\sum_{1}^{\infty}\frac{b_n}{n^2}.
\end{aligned}
$$

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1) A. Kolmogoroff, loc. cit.

2) P. R. Halmos, loc. cit.

Hence by Theorem 1, we get our theorem.

We note here that A. Kolmogoroff made use of his inequality

$$P_r(\max_{1\leq i\leq n} |s_i| \geqq nR) \leqq \frac{1}{n^2R^2}\sum_{i=1}^{n} b_i ,$$

which is rather deeper than Tchebycheff inequality.