16. Weak Topology, Bicompact Set and the Principle of Duality.

By Shizuo KAKUTANI.
Mathematical Institute, Osaka Imperial University.
(Comm. by T. TAKAGI, M.I.A., March 12, 1940.)

1. *Weak topology and bicompact set.* Let $\mathfrak{S}$ be an abstract space and consider the family Ω of real-valued (bounded) functions $f(x)$ defined on $\mathfrak{S}$. We shall introduce a weak topology on Ω. For any $f_0(x)\in\Omega$ its weak neighbourhood $U(f_0; x_1, x_2, \dots, x_n; \varepsilon)$ is defined as the totality of all the functions $f(x)\in\Omega$ such that $|f(x_i)-f_0(x_i)|<\varepsilon$ for $i=1,2,\dots,n$, where $\{x_i\}(i=1,2,\dots,n)$ is an arbitrary finite system of points from $\mathfrak{S}$, and $\varepsilon>0$ is an arbitrary positive number. It is clear that this neighbourhood system defines a topology on Ω. This topology is called the *$\mathfrak{S}$-weak topology of Ω as functionals.*

In the same way, if we consider, for any $x_0\in\mathfrak{S}$, $x_0(f)\equiv f(x_0)$ as a real-valued (bounded) function defined on Ω, we can introduce a weak topology on $\mathfrak{S}$. Indeed, for any $x_0\in\mathfrak{S}$ its weak neighbourhood $U(x_0; f_1, f_2, \dots, f_n; \varepsilon)$ is defined as the totality of all the points $x\in\mathfrak{S}$ such that $|f_i(x)-f_i(x_0)|<\varepsilon$ for $i=1,2,\dots,n$, where $\{f_i(x)\}(i=1,2,\dots,n)$ is an arbitrary finite system of functions from Ω, and $\varepsilon>0$ is an arbitrary positive number. It is again clear that this neighbourhood system defines a topology on $\mathfrak{S}$. This topology is called the *Ω-weak topology of $\mathfrak{S}$ as points.*

Theorem 1. If Ω is the totality of all the bounded (not necessarily continuous) real-valued functions $f(x)$ defined on $\mathfrak{S}$ such that $|f(x)|\leqq 1$ for any $x\in\mathfrak{S}$, then $\mathfrak{S}$ is bicompact with respect to the $\mathfrak{S}$-weak topology of Ω as functionals.

This theorem is due to A. Tychonoff.[1] The weak topologies of the same kind may also be defined analogously even if the range of $f(x)$ is contained in an arbitrary uniform space (in the sense of A. Weil).[2] In the special case, when $\mathfrak{S}$ is a Banach space E and Ω is the set of all the bounded linear functionals $f(x)$ defined on E (i. e., $\Omega=\bar{E}$), these weak topologies become the usual ones. We have once[3] studied the weak topologies of Banach spaces, and have shown that these weak topologies are useful in the problems concerning the regularity of Banach spaces. From Theorem 1 we can easily deduce

Theorem 2. Let E be a Banach space. Then the unit sphere: $\|f\|\leqq 1$ of the conjugate space $\bar{E}$ of E is bicompact with respect to the E-weak topology of $\bar{E}$ as functionals.

---

1) A. Tychonoff: Über einen Funktionenraum, Math. Ann., **111** (1935), 762–766.

2) A. Weil: Sur les espaces à structure uniforme et sur la topologie générale, Actualité, 551, Paris, 1937.

3) S. Kakutani: Weak topology and regularity of Banach spaces, Proc. **15** (1939), 169–173.

Combined with a result of E. Helly,[1] this theorem gives

Theorem 3. In order that a Banach space E is regular (=reflexive), it is necessary and sufficient that the unit sphere: $\|x\| \leqq 1$ *of E is bicompact with respect to the* $\bar{E}$*-weak topology of E as points.*

2. *Principle of duality.* It is easy to see that in the theory of L. Pontrjagin[2] and E. R. van Kampen[3] concerning the duality of topological commutative groups, the topology of character group is nothing but the weak topology in the sense given in §1. Thus, weak topologies are useful in the theory of duality. Indeed, the theory of regular Banach spaces is itself a kind of duality of theorems. In the following lines we shall show that the same weak topologies are useful in the theory of duality of a more general character.

Let us consider a (topological) space G with relations between its elements (group, ring, linear space, lattice, etc.), and we shall try to represent G isomorphically (and homeomorphically) by a concrete set of functions defined on some bicompact topological space Ω. Our fundamental principle may be summed up as follows:

Let G_0 *be the simplest non-trivial space of the same kind (i. e., the unit circle K in the case of topological commutative groups; the ring* M_n *of all finite n-dimensional matrices in the case of non-commutative groups or rings; (ordered) one-dimensional Euclidean space* R_1 *in the case of (semi-ordered) Banach spaces; the Baolean ring* $\mathfrak{B}_0$ *composed of only* 0 *and* 1 *in the case of Boolean rings; and the n-dimensional projective geometry* P_n *in the case of modular lattices, etc.), and consider the totality* Ω *of all the (bounded continuous) homomorphisms* $f(x)$ *which maps G into* G_0 *(satisfying some normalizing conditions). Then* Ω *is bicompact with respect to the G-weak topology of* Ω *as functionals, and* $x(f) \equiv f(x)$ *may be considered as a bounded continuous function defined on* Ω *whose range lies in* G_0. *Under some additional conditions, G is isomorphically (and homeomorphically) represented by the system of these functions* $x(f)$ *defined on* Ω.

3. *Concrete representation of Boolean rings.* As an example, we shall first discuss the concrete representation of abstract Boolean rings. The well-known result of M. H. Stone[4] and H. Wallman[5] reads as follows:

Theorem 4. For any abstract Boolean ring $\mathfrak{B}$ *with unit, there exists a totally disconnected bicompact topological space* Ω *such that the*

---

1) E. Helly: Über Systeme linearer Gleichungen mit unendlich vielen Unbekannten, Monatshefte für Math. u. Phys., **31** (1921), 60-91.

2) L. Pontrjagin: The theory of topological commutative groups, Annals of Math., **35** (1934), 361-388. See also: L. Pontrjagin, Topological groups, Princeton, 1939.

3) E. R. van Kampen: Locally bicompact abelian groups and their character groups, Annals of Math., **36** (1935), 448-463.

4) M. H. Stone: Applications of the theory of Boolean rings to general topology, Trans. Amer. Math. Soc., **41** (1937), 375-481.

M. H. Stone: The theory of representations for Boolean algebras, Trans. Amer. Math. Soc., **40** (1936), 37-111.

5) H. Wallman: Lattices and topological spaces, Annals of Math., **39** (1938), 112-126.

concrete Boolean ring of all the open-and-closed subsets of Ω is isomorphic to the given $\mathfrak{B}$.

Let $\mathfrak{B}$ be an abstract Boolean ring with unit 1, and consider the totality Ω of all the homormorphic mappings $f(x)$ of $\mathfrak{B}$ on the Boolean ring $\mathfrak{B}_0$ composed of only 0 and 1. In other words, Ω is the totality of all real-valued functions $f(x)$ defined on $\mathfrak{B}$ such that $f(x)=0$ or $=1$ for any $x \in \mathfrak{B}$ and which satisfy the following conditions:

$$(1)\quad f(x \vee y)=f(x) \vee f(y),$$
$$(2)\quad f(x \wedge y)=f(x) \wedge f(y),$$
$$(3)\quad f(x+y)=f(x)+f(y), \quad (\text{mod. } 2),$$
$$(4)\quad f(x \cdot y)=f(x) \cdot f(y),$$
$$(5)\quad f(0)=0, \qquad f(1)=1.$$

(These conditions are clearly not mutually independent). Ω is clearly bicompact with respect to the $\mathfrak{B}$-weak topology of Ω, and, as is easily seen, $\mathfrak{B}$ is homomorphically represented by the system of real-valued continuous functions $x(f) \equiv f(x)$ defined on Ω. In order to show that this is even an isomorphic representation, we have only to show that there exists, for any $a \in \mathfrak{B}$, $b \in \mathfrak{B}$, $a \neq b$, a function $f(x) \in \Omega$ such that $f(a) \neq f(b)$. This fact corresponds to the existence of a prime ideal, which contains the one but not the other, in the proof of M. H. Stone. This was proved by him by appealing to the transfinite induction. We can, however, prove this by using the notions of bicompact sets (without appealing to the transfinite method; although these are essentially the same). Indeed, let A be any finite subset of $\mathfrak{B}$ which contains 0, 1, a, and b. Then the Boolean subring $\mathfrak{B}(A)$ of $\mathfrak{B}$ generated by A is composed of finite elements, and there exists clearly a homomorphic mapping $f(x)$ of $\mathfrak{B}(A)$ on $\mathfrak{B}_0$ such that $f(a) \neq f(b)$ and which satisfies the conditions (1)–(5) for any $x, y \in \mathfrak{B}(A)$. This follows directly from the fact that Theorem 4 is trivially true for a Boolean ring with only finite elements. Let $\Omega(A)$ be the set of all the real-valued functions $f(x)$ defined on $\mathfrak{B}$ whose value $f(x)$ is 0 or 1, and which satisfy the condition (1)–(5) for any $x, y \in \mathfrak{B}(A)$ ($f(x)=0$ or $=1$ arbitrarily on $\mathfrak{B} - \mathfrak{B}(A)$). By the above result, $\Omega(A)$ is not empty for any finite subset A of $\mathfrak{B}$, and it will be clear that $\Omega(A)$ is bicompact with respect to the $\mathfrak{B}$-weak topology of $\Omega(A)$ as functionals. Moreover, for any finite system $\{A_i\} (i=1, 2, \ldots, n)$ of finite subsets of $\mathfrak{B}$, we have $\prod_{i=1}^{n} \Omega(A_i) \neq 0$. This follows directly from the fact that we have $\prod_{i=1}^{n} \Omega(A_i) \supset \Omega(\sum_{i=1}^{n} A_i) \neq 0$. Hence, by the well-known property of bicompact sets, the totality of all $\Omega(A)$ (for all finite subsets A of $\mathfrak{B}$) has at least one common element $f(x)$, and this $f(x)$ is clearly a homomorphic mapping of $\mathfrak{B}$ on $\mathfrak{B}_0$ with the required properties.

Thus we have proved that the concrete representation obtained above is an isomorphism. Let now $E(a)$ be the set of all $f(x) \in \Omega$ such that $f(a)=1$. Then $E(a)$ is clearly open-and-closed in Ω, and $\mathfrak{B}$ is isomorphic to the concrete Boolean ring of all such sets $E(a)$, $a \in \mathfrak{B}$.

Since it is clear that conversely any open-and-closed subset E of Ω corresponds to some $a \in \mathfrak{B} : E = E(a)$, the proof of Theorem 4 is completed.

4. *Concrete representation of abstract semi-ordered Banach spaces.* Next we shall discuss the concrete representation of abstract semi-ordered Banach spaces. As an example we shall treat the case of an abstract (M)-space. A semi-ordered Banach space is called to be an *abstract* (M)*-space* if it satisfies, besides the ordinary axioms of semi-ordered Banach spaces, the condition:

$$(^*) \quad x \geqq 0 \quad \textit{and} \quad y \geqq 0 \quad \textit{imply} \quad \|x \vee y\| = \max(\|x\|, \|y\|).$$

This is a Banach space which is dual to the *abstract* (L)*-space*, for which the condition $(^*)$ is substituted by[1)]

$$(^{**}) \qquad x \geqq 0 \quad \textit{and} \quad y \geqq 0 \quad \textit{imply} \quad \|x+y\| = \|x\| + \|y\|.$$

Indeed, it will be easily seen that the conjugate space of an abstract (L)-space is an abstract (M)-space, and that conversely the conjugate space of an abstract (M)-space is an abstract (L)-space. But these spaces are in general (i. e., except the finite-dimensional case) not regular. We have proved in another paper[2)] the following

Theorem 5. For any abstract (L)*-space* (AL) *with unit, there exists a totally disconnected bicompact topological space* Ω *and a completely additive measure defined on the Borel sets of* Ω*, such that* (AL) *is isometric and lattice-isomorphic to the concrete* (L)*-space of all the integrable functions* $x(t)$ *defined on* Ω *(with* $\|x\| = \int_{\Omega} |x(t)|\, dt$ *as its norm and with* $x \geqq y : x(t) \geqq y(t)$ *almost everywhere on* Ω*, as its semi-ordering).*

Hereby, under the unit element of an abstract (L)-space (AL), we understand the positive element 1 of (AL) such that $x \wedge 1 > 0$ for any $x > 0$.

We shall here prove

Theorem 6. For any abstract (M)*-space* (AM) *with unit, there exists a bicompact topological space* Ω *such that* (AM) *is isometric and lattice-isomorphic to the concrete* (M)*-space* $C(\Omega)$ *of all the continuous functions* $x(t)$ *defined on* Ω *(with* $\|x\| = \max_{t \in \Omega} |x(t)|$ *as its norm and* $x \geqq y : x(t) \geqq y(t)$ *for any* $t \in \Omega$*, as its semi-ordering).*

Hereby, under the unit element of an abstract (M)-space (AM), we understand the positive element 1 of (AM) such that $x \leqq 1$ for any x with $\|x\| \leqq 1$.

In order to prove this theorem, let Ω be the totality of all the bounded linear functionals $f(x)$ defined on (AM) such that

$$(1) \quad f(x \vee y) = f(x) \vee f(y),$$
$$(2) \quad f(x \wedge y) = f(x) \wedge f(y),$$

1) S. Kakutani: Mean ergodic theorem in abstract (L)-spaces, Proc. **15** (1939), 121–123.

2) S. Kakutani: Concrete representation of abstract (L)-spaces and the mean ergodic theorem, forthcoming.

(3) $f(x+y)=f(x)+f(y)$,

(4) $f(\lambda x)=\lambda f(x)$, $\quad \lambda \gtreqless 0$,

(5) $f(1)=1$,

(6) $f(x) \leqq \|x\|$

for any $x, y \in (AM)$. (These conditions are clearly not mutually independent). Ω is clearly bicompact with respect to the (AM)-weak topology of Ω as functionals, and each $x(f) \equiv f(x)$ is continuous on Ω. Moreover, we have $\max_{f \in \Omega} |x(f)| \leqq \|x\|$.

Thus (AM) is continuously and homomorphically represented by the system of these functions $x(f)$ defined on Ω. In order to prove that this representation is even isometric and isomorphic, we have only to prove that for any $x_0 > 0$ there exists an $f_0(x) \in \Omega$ such that

(7) $f_0(x_0) = \|x_0\|$.

The proof of this fact is not so easy, but may be carried out in an elementary way.

Thus we have proved that (AM) is isometrically and lattice-isomorphically represented by the system of functions $x(f)$ defined on Ω. Since it will be easily seen that conversely any continuous function $x(f)$ defined on Ω is contained in the this system, the proof of Theorem 6 is completed.

The detail of the proofs of these theorems and the discussion of the allied problems will be published in another paper.