PAPERS COMMUNICATED

37. *A Remark on the Non-vanishing of Almost Periodic Functions.*

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1. Concerning the non-vanishing of an almost periodic function I have proved the following theorem[1]:

Theorem 1. Let $f(x)$ be S^2 a.p. and its Fourier exponents greater than some number A have gaps tending to infinity. If $f(x)=0$ almost everywhere in some interval, then $f(x)=0$ almost everywhere in $(-\infty,\infty)$.

This will be generalized in the following form, the proof of which is quite similar as that of Theorem 1 and then I do not repeat it here[2].

Theorem 2. Suppose that the conditions in Theorem 1 are satisfied. Let $g(z)=g(x+iy)$ $(z=x+iy)$ be a function analytic in $c<x<d$, $0<y<r$ and continuous in $c<x<d$, $0\leqq y<r$. Then if $f(x)=g(x)$ almost everywhere in some interval in (c, d), then $f(x)=g(x)$ almost everywhere in (c, d).

In the theory due to Levinson, Paley and Wiener of non-vanishing of purely periodic functions or functions defined in $(-\infty,\infty)$, some kind of integrability being assumed, two types of theorems are known[3].

In theorems of the first type it is supposed that the Fourier coefficients or Fourier transform tends to zero with considerable rapidity, and in the second type it is assumed among other conditions that the Fourier coefficients have gaps which increase indefinitely or the breadth of intervals in which the Fourier transform vanishes tends to infinity[4].

Plainly Theorem 1 and 2 are of the second type. Thus it will be desirable to obtain a theorem of the first type for the class of almost periodic functions. In the present paper I shall prove such a theorem (Theorem 3) by reducing it to a fundamental result concerning ordinary Fourier transforms which is due to N. Levinson[5]. It runs as follows:

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1) T. Kawata, A gap theorem for the Fourier series of an almost periodic function, Tôhoku Math. Journ., **43** (1937).

2) T. Kawata, Non-vanishing of functions and related problems, Tôhoku Math. Journ., **46** (1940).

3) Paley and Wiener, Fourier transforms in the complex domain, Amer. Colloq. 19, pp. 123–127. pp. 116–121.

N. Levinson, On a class of non-vanishing functions, Proc. London Math. Soc., (2) **41** (1936).

N. Levinson, A theorem relating non-vanishing and analytic functions, Journ. Math. and Phys., **16** (1938).

T. Kawata, loc. cit. Non-vanishing etc.

T. Kawata, A theorem concerning non-vanishing of functions, Proc. **14** (1938).

4) T. Kawata, loc. cit. Non-vanishing etc.

5) N. Levinson, loc. cit. A theorem relating non-vanishing etc.

Theorem A. *Suppose that $f(x) \in L_2(-\infty, \infty)$ and its Fourier transform $F(x)$ satisfies that*

$$F(x) = O\Big(\exp\big(-\theta(x)\big)\Big), \tag{1}$$

as $x \to \infty$, where $\theta(x)$ is an increasing function for $x > 0$ such that

$$\int_1^\infty \frac{\theta(x)}{x^2} dx = \infty . \tag{2}$$

Then if $f(x) = g(x)$ almost everywhere in some interval in (c, d) where $g(z)$ $(z = x + iy)$ is a function analytic in $c < x < d$, $0 < y < r$ and in $c < x < d$, $0 \leqq y < r$, then $f(x) = g(x)$ holds almost everywhere in (c, d).

2. We write

$$D\{f\} = \underset{-\infty < x < \infty}{\text{u. b.}} \left\{\int_x^{x+1} |f(x)|^2 dx\right\}^{\frac{1}{2}}.$$

Then if there exists a sequence of trigonometrical polynomials $f_n(x)$ such that

$$\lim_{n \to \infty} D\{f_n - f\} = 0 , \tag{3}$$

then $f(x)$ is said to be S^2. a. p.

Theorem 3. *Let $f(x)$ be S^2. a. p. and its Fourier series be*

$$\sum a_n e^{i\lambda_n x}. \tag{4}$$

Let there exists an N such that

$$A(u) = \sum_{-u - \frac{1}{2} < \lambda_k < -u + \frac{1}{2}} |a_k| \tag{5}$$

be finite for each u $(\geqq N)$ and

$$A(u) = O\Big(\exp\big(-\theta(u)\big)\Big), \tag{6}$$

where $\theta(u)$ satisfies (2). Further let $g(z)$ $(z = x + iy)$ be a function analytic in $c < x < d$, $0 < y < r$ and continuous in $c < x < d$, $0 \leqq y < r$. Then if $f(x) = g(x)$ almost everywhere in some interval in (c, d), then it holds almost everywhere in (c, d).

Lemma 1. *Let $f(x)$ be S^2. a. p. and*

$$\lim_{n \to \infty} D\{f_n - f\} = 0 , \tag{7}$$

where $f_n(x)$ are S^2. a. p.

Then $f(x)$ $\sin^2 \frac{1}{4} x / x^2$ belongs to $L_p(-\infty, \infty)$ for every $1 \leqq p \leqq 2$, and

$$\lim_{n \to \infty} \int_{-\infty}^\infty e^{iux} f_n(x) \frac{\sin^2 \frac{1}{4} x}{x^2} dx = \int_{-\infty}^\infty e^{iux} f(x) \frac{\sin^2 \frac{1}{4} x}{x^2} dx . \tag{8}$$

It is evident by assumptions, that $D\{f_n\} < K$, K being a constant independent of n. We have

$$\left|\int_N^\infty e^{iux}\{f_n(x)-f(x)\}\frac{\sin^2\frac{1}{4}x}{x^2}dx\right|$$

$$\leqq\sum_{k=N}^\infty\int_k^{k+1}|f_n(x)-f(x)|\frac{\sin^2\frac{1}{4}x}{x^2}dx$$

$$\leqq\sum_{k=N}^\infty\left\{\int_k^{k+1}|f_n(x)-f(x)|^2dx\right\}^{\frac{1}{2}}\left\{\int_k^{k+1}\frac{\sin^4\frac{1}{4}x}{x^4}dx\right\}^{\frac{1}{2}}$$

$$\leqq K\sum_{k=N}^\infty\frac{1}{k^2}<\varepsilon\,,$$

for $N>N_0(\varepsilon)$. The same is true for $\int_{-\infty}^{-N}$. Similarly we have

$$\lim_{n\to\infty}\left|\int_{-N}^{N}e^{+iux}\{f_n(x)-f(x)\}\frac{\sin^2\frac{1}{4}x}{x^2}dx\right|$$

$$\leqq\lim_{n\to\infty}D\{f_n-f\}\cdot\left(\frac{1}{2}+\sum_{k=1}^{N-1}\frac{1}{k^2}\right)=0\,.$$

Thus $f(x)\sin^2\frac{1}{4}x/x^2$ belongs to $L_1(-\infty,\infty)$ and (8) holds. That it belongs to $L_p(-\infty,\infty)$ $(1\leqq p\leqq 2)$ is also immediate.

Lemma 2.[1] *There exists a sequence of trigonometrical polynomials* $\{f_n(x)\}$ *satisfying* (7) *and such that if*

$$f_n(x)=\sum_1^n a_k^{(n)}e^{i\lambda_k x}\,,$$

$$f(x)\sim\sum a_k e^{i\lambda_k x}\,,$$

then

$$a_k=\lim_{n\to\infty}a_k^{(n)}\,.$$

Proof of Theorem 3. Let $\Phi(u)$ and $\Phi_n(u)$ be the Fourier transforms of $f(x)\sin^2\frac{1}{4}x/x^2$ and $f_n(x)\sin^2\frac{1}{4}x/x^2$ respectively. Then we have

$$\Phi_n(u)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{+iux}\sum_{k=1}^n a_k^{(u)}e^{i\lambda_k x}\frac{\sin^2\frac{1}{4}x}{x^2}dx$$

$$=\frac{1}{\sqrt{2\pi}}\sum_{k=1}^\infty a_k^{(n)}\int_{-\infty}^\infty e^{i(u+\lambda_k)x}\frac{\sin^2\frac{1}{4}x}{x^2}dx\,.$$

1) Well-known.

Since $\int_{-\infty}^{\infty} e^{+ivx} \sin^2 \frac{1}{4} x/x^2 dx = 0$ except in $-1/2 < v < 1/2$, we have

$$|\Phi_n(u)| \leqq \sum_{-u-\frac{1}{2}<\lambda_k<-u+\frac{1}{2}} |a_k^{(n)}|.$$

By Lemmas 1 and 2, we have

$$|\Phi(u)| \leqq \sum_{-u-\frac{1}{2}-\lambda_k<-u+\frac{1}{2}} |a_k| = O(e^{-\theta(u)}) \quad \text{for} \quad u \to \infty.$$

Hence by Theorem A, our assertion is immediate.