83. *On the Theory of Spectra.*

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The "algebraization" of the spectral theory, inaugurated by J. von Neumann, H. Freudenthal and S. Steen, was taken up recently by S. Kakutani, F. Riesz, M. H. Stone and B. Vulich,[1] and was treated with their respective methods and results. The purpose of the present note is to give a ring-lattice-theoretic treatment of the problem, stressing the analogy to the field of real numbers. Without assuming metrical (even topological) nor divisibility axiom, a *characterisation of the function ring of the Borel-measurable functions*[2] is obtained. Thus the results may be applied to the operator theory as well as to the theory of probability.

§ 1. *Axioms of Pythagorean ring.* A system $\mathfrak{R}$ of elements $A, B, \dots, X, Y, Z$ is called a "Pythagorean ring" if it satisfies the following axioms.

(A–1) $\mathfrak{R}$ is a commutative, associative ring with unit I, admitting the field of real numbers as coefficients (Operatoren).—The real numbers will be denoted by small greak letters.

(A–2) $X^2=0$ implies $X=0$.

(A–3) If non-zero element X of the form $X=Y^2$ is called "positive" (in symbol $X>0$), then the sum of positive element and "non-negative" element is positive, viz. $X^2 \neq 0$ or $Y^2 \neq 0$ implies the existence of $Z^2 \neq 0$ such that $X^2+Y^2=Z^2$.

(A–4) By the semi-order relation $X>Y(X-Y>0)$, there exists, for all X, the lowest upper bound (l. u. b.) $\sup(X, 0)$ of X and 0.—We will write $\sup(X, 0)=X^+$, $\sup(-X, 0)=X^-$ and $|X|=X^+ + X^-$.

(A–5) $X^+ \cdot X^- = 0$ for all X.

(A–6) Monotone increasing sequence $\{X_n\}$ bounded from above admits the l. u. b. $\sup\limits_{n\geqq 1} X_n$.

(A–7) If $A>0$, $X_i \geqq 0$, $X_{i+1} \geqq X_i$ and $\sup\limits_{i\geqq 1} X_i$ exists, then $A \cdot \sup\limits_i X_i = \sup\limits_i (A \cdot X_i)$.

Remark 1. The "real" character of $\mathfrak{R}$ is expressed by the "Pythagorean axiom" (A–3) together with (A–2). (A–4) and (A–6) are lattice-theoretic axioms.[3] (A–5) is equivalent to $|X^2|=|X|^2$, and (A–7) means a generalised distributive law.

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1) J. von Neumann: Rec. Math., **43** (1936), 415–484. H. Freudenthal: Proc. Akad. Amsterdam, **39** (1936), 641–651. S. Steen: Proc. London Math. Soc., **41** (1936), 361–392. S. Kakutani: Proc. **15** (1939), 121–123. F. Riesz: Ann. Math., **41** (1940), 174–206. M. H. Stone: Proc. Nat. Acad. Sci., **26** (1940), 280–283. B. Vulich: C. R. URSS, **26** (1940), 850–859. The last two papers appeared during the preparation of the present note. In the redaction, the writer is much suggested by Steen's paper.

2) Not necessarily bounded!

3) Concerning lattice see G. Birkhoff: Lattice Theory, New York (1940).

Remark 2. If we assume the "boundedness axiom," i.e. for any X, there exist $\alpha, \beta \geqq 0$ with $-\alpha I \leqq X \leqq \beta I$, then it is easy to see that the axioms (A-5) and (A-7) are redundant. In this case we may replace (A-3) and (A-4) by the postulate of the existence of semi-ordering satisfying (2-1) below. For we are able to make use of the power series expansion of "square root." Thus, in this bounded case, our results coincides with that of M. H. Stone. The details and applications will be published elsewhere.

§ 2. *Some preliminary consequences from the axioms.*

(2-1) $I > 0$. $X > Z$, $Y \geqq W$ and $\alpha \gtrless 0$ imply $X + Y > Z + W$ and $\alpha X \gtrless \alpha Z$. If $X > Z > 0$, $Y \geqq W \geqq 0$, then $XY \geqq ZW \geqq 0$. $X^2 > 0$ if $X \neq 0$.

(2-2) By the semi-order relation $X > Y$, $\mathfrak{R}$ is a lattice, viz. to any pair X, Y there corresponds the l. u. b. $\sup(X, Y) = (X-Y)^+ + Y = (Y-X)^+ + X$ and the greatest lower bound (g. l. b.) $\inf(X, Y) = -\sup(-X, -Y)$.

Proof. By (2-1), the translations $A \to A+B$ and the expansions $A \to \alpha A$ $(\alpha > 0)$ induce one-one transformations of $\mathfrak{R}$ which preserve the semi-ordering; the one-one transformation $A \to -A$ of $\mathfrak{R}$ inverts the semi-ordering.

(2-3) $\sup(X+Z, Y+Z) = \sup(X, Y) + Z$.

(2-4) $\sup(X, Y) + \inf(X, Y) = X + Y$, in particular $X = X^+ - X^-$.

Proof. Add X, Y to $\sup(X, Y) + \inf(-X, -Y) = 0$ and apply (2-3).

(2-5) The pair (X^+, X^-) is characterised by $X = X^+ - X^-$, $X^+ \cdot X^- = 0$, $X^+ \geqq 0$ and $X^- \geqq 0$.

Proof. Let $X = X' - X''$, $X' \cdot X'' = 0$, $X' \geqq 0$, $X'' \geqq 0$, then $(X^+ - X')^2 = (X^+ - X') \cdot (X^- - X'') = -(X^+ \cdot X'' + X' \cdot X^-)$. Thus $(X^+ - X')^2 \leqq 0$, and so $X^+ = X'$ by (A-2).

(2-6) $A \cdot \sup(X, Y) = \sup(A \cdot X, A \cdot Y)$ $\big(A \cdot \inf(X, Y) = \inf(A \cdot X, A \cdot Y)\big)$, if $A \geqq 0$.

Proof. By (2-3), it is sufficient to prove the case $Y = 0$, i.e. $A \cdot X^+ = (A \cdot X)^+$. Now $AX = AX^+ - AX^-$, $AX^+ \geqq 0$, $AX^- \geqq 0$, $AX^+ \cdot AX^- = A^2 \cdot (X^+ \cdot X^-) = 0$, and hence $A \cdot X^+ = (A \cdot X)^+$ by (2-5).

(2-7) Any sequence $\{X_n\}$ bounded from above (below) admits the l. u. b. $\sup\limits_{n \geqq 1} X_n$ (g. l. b. $\inf\limits_{n \geqq 1} X_n$).

Proof. Put $X'_n = \sup\limits_{n \geqq m} X_m$ and apply (A-6) to $\{X'_n\}$.

(2-8) If $\sup\limits_{n \geqq 1} X_n$, $\sup\limits_{n \geqq 1} Y_n$ $(\inf\limits_{n \geqq 1} X_n, \inf\limits_{n \geqq 1} Y_n)$ exist, then

$$\sup_{n,m}(X_n + Y_m) = \sup_n X_n + \sup_n Y_n \ \Big(\inf_{n,m}(X_n + Y_m) = \inf_n X_n + \inf_n Y_n\Big).$$

Proof. Surely $\sup(X_n + Y_m) \leqq \sup X_n + \sup Y_n$. Let the equality does not hold good, then $\sup X_n \nleqq \sup(X_n + Y_m) - \sup Y_n$. Thus X_n exists such that $X_n \nleqq \sup(X_n + Y_m) - \sup Y_n$, and so Y_m exists such that $X_n + Y_m \nleqq \sup(X_n + Y_m)$, which is a contradiction.

(2-9) If $X_n \geqq 0$, $Y_n \geqq 0$ and $\sup\limits_{n \geqq 1} X_n$ and $\sup\limits_{n \geqq 1} Y_n$ $(\inf\limits_{n \geqq 1} X_n, \inf\limits_{n \geqq 1} Y_n)$

exist, then $(\sup X_n)\cdot(\sup Y_n)=\sup(X_n\cdot Y_m)$ $\big((\inf X_n)\cdot(\inf Y_n)=\inf (X_n\cdot Y_m)\big)$.

Proof. By (A-7) and (2-6), we have $X_i\cdot\sup_n \sup_{n\ge n} Y_m=\sup_n \sup_{n\ge m} (X_i\cdot Y_m)=\sup_n (X_i\cdot Y_n)$. Thus we have $\sup_{n,m}(X_n\cdot Y_m)=\sup_n (X_n\cdot \sup_m Y_m)=(\sup X_n)\cdot(\sup Y_n)$

(2-10) Let $X>0$, then $\inf_{n\ge1}(X/n)=0$.

Proof. Let $\inf(X/n)=Y>0$, then $0<\sup_n(nY)\leqq X$. By $Y>0$ we have $\sup(nY)\nleqq \sup(nY)-Y$ and hence $mY\nleqq\sup(nY)-Y$ for an m. Thus $(m+1)Y\nleqq \sup(nY)$, which is a contradiction.

(2-11) Let $0<X$, $0<Y$, then $X^2<Y^2$ implies $X<Y$.

Proof. $(Y-X)(Y+X)>0$ and hence $(Y-X)^-\cdot(Y-X)(Y+X)=-\big((Y-X)^-\big)^2\cdot(Y+X)\geqq0$ by (A-5). Thus $\big((Y-X)^-\big)^2\cdot(Y+X)=0$, and hence by (A-3) and (2-1), we obtain $\big((Y-X)^-\big)^2\cdot Y=0$, $\big((Y-X)^-\big)^2\cdot X=0$. Thus $\big((Y-X)^-\big)^2\cdot(Y-X)=-\big((Y-X)_-\big)^3=0$ and so $(Y-X)^-=0$ by (A-2). Therefore we must have $Y>X$.

§ 3. *Spectral theorem for positive elements.*

(3-1) Let $X>0$, then $X=Y^2$. By (A-5) we have $(Y^++Y^-)^2=(Y^+)^2+(Y^-)^2=(Y^+-Y^-)^2=Y^2=X$. We call (Y^++Y^-) the "positive square root" $X^{\frac12}$ of X. Because of (2-11), $X^{\frac12}$ is characterised by $X^{\frac12}\geqq0$, $(X^{\frac12})^2=X$.

(3-2) Let $X>0$ and put $X_1=\inf_{n\ge1}X^n$, $X_2=X-X_1$. Then $X_1^2\geqq X_1$, $X_1\cdot X_2=0$ and $X_2^2\leqq X_2$.

Proof. By (2-9) $X_1^2=\inf_{n,m}X^{n+m}\geqq X_1$, $X_1\cdot X_2=X_1\cdot X-X_1^2=\inf_n X^{n+1}-\inf_{n,m}X^{n+m}=0$. From $X^{n+1}-X^2-X^n+X=X(X-I)^2\cdot(X^{n-2}+X^{n-3}+\cdots+I)\geqq0$ we obtain $\inf_n X^{n+1}-X^2\geqq\inf_n X^n-X$, i. e. $XX_1-X^2=X_1^2-X^2\geqq X_1-X$. This proves $X_2^2\leqq X_2$ by $X_2^2=X^2-2XX_1+X_1^2=X^2-X_1^2$.

(3-3) By (3-2) and (2-11) we have $X_1\geqq X_1^{\frac12}\geqq X_1^{\frac14}=(X^{\frac12})^{\frac12}\geqq\cdots\geqq X_1^{\frac{1}{2n}}\geqq\cdots\geqq0$. Thus, by (2-7) and (2-9), $\inf_{n\ge1}X_1^{\frac{1}{2n}}=E$ exists and $E^2=E\leqq X_1$.

(3-4) $XE=X_1=X_1E\geqq E$, $X(I-E)=X_2=X_2(I-E)\leqq(I-E)$.

Proof. $X_2E=X_2\cdot\inf X_1^{\frac{1}{2n}}=\inf(X_2X_1^{\frac{1}{2n}})=\inf\big((X_2^{2^n}\cdot X_1)^{\frac{1}{2n}}\big)=\inf\big((X_2^{2^n-1}\cdot X_2X_1)^{\frac{1}{2n}}\big)=0$. Thus $XE=X_1E+X_2E=X_1\cdot E$. Next from $X_1\leqq X_1^2$ we obtain $X_1^{2^n}\leqq X_1^{2^n+1}$ and hence $X_1\leqq\inf(X_1X_1^{\frac{1}{2n}})=X_1\cdot\inf X_1^{\frac{1}{2n}}=X_1E$. On the other hand $E\leqq I$ by $I-E=I-2E+E^2=(I-E)^2\geqq0$ and thus $X_1\geqq X_1E$. Therefore $X_1=X_1E$. We have already $X(I-E)=X-X_1=X_2=X_2(I-E)$. To show that $X_2(I-E)\leqq I-E$, it is sufficient to prove $X_2\leqq I$. Now we have $0\leqq(I-X_2)^2=I-2X_2+$

$X_2^2 \leqq I-2X_2+X_2=I-X_2$ by $X_2^2 \leqq X_2$, i.e. $X_2 \leqq I$.

(3–5) For any $X>0$ and $\lambda>0$, put

$$E_\lambda = I - \inf_{m\geq 1}\left(\inf_{n\geq 1}(X/\lambda)^n\right)^{\frac{1}{2m}}.$$

Then, by (3–4), $E_\lambda^2=E_\lambda$, $E_\lambda\cdot X=E_\lambda\cdot(X/\lambda)\lambda \leqq \lambda E_\lambda$, $X(I-E_\lambda)=\lambda(X/\lambda)\cdot(I-E_\lambda)\geqq \lambda(I-E_\lambda)$.

(3–6) If $\lambda>\mu>0$, we have $E_\mu \leqq E_\lambda$ and $E_\mu\cdot E_\lambda=E_\mu$, i.e. $(E_\lambda-E_\mu)^2=(E_\lambda-E_\mu)$.

Proof. That $E_\lambda \geqq E_\mu$ is evident, and hence $E_\mu=E_\mu^2\leqq E_\mu\cdot E_\lambda$. We have, by $E_\lambda\leqq I$, $E_\mu\cdot E_\lambda\leqq E_\mu$ and thus $E_\mu\cdot E_\lambda=E_\mu$.

(3–7) Let $\lambda>\mu>0$, then $\mu(E_\lambda-E_\mu)\leqq X(E_\lambda-E_\mu)\leqq\lambda(E_\lambda-E_\mu)$.

Proof. $X(E_\lambda-E_\mu)=XE_\lambda(E_\lambda-E_\mu)=XE_\lambda(I-E_\mu)\geqq E_\lambda\big(\mu(I-E_\mu)\big)=\mu(E_\lambda-E_\mu)$ by (3–6). The last inequality may be proved in the same manner.

(3–8) If we put $\inf\limits_{\lambda>0}E_\lambda=E_{+0}$ and $\sup\limits_{\lambda>0}E_\lambda=E_\infty$, then $XE_{+0}=0$ and $E_\infty=I$.

Proof. $XE_{+0}=0$ follows from $0\leqq XE_\lambda\leqq\lambda E_\lambda$. We have $X/\lambda\geqq(I-E_\lambda)\geqq 0$. Thus, by (2–10), $E_\infty=I$.

(3–9) $E_\lambda=E_{\lambda-0}=\sup\limits_{\lambda>\mu}E_\mu$.

Proof. We have $\lambda(E_\lambda-E_{\lambda-0})=X(E_\lambda-E_{\lambda-0})$. Hence, by the definition of E_λ and the idempotent character of $(E_\lambda-E_{\lambda-0})$, $(E_\lambda-E_{\lambda-0})=(X/\lambda)^n\cdot(E_\lambda-E_{\lambda-0})=(I-E_\lambda)\cdot(E_\lambda-E_{\lambda-0})=0$.

(3–10)—*Spectral theorem.* Let $X>0$, $\mu>0$, $0=\lambda_0<\lambda_1<\lambda_2<\cdots<\lambda_k=\mu$ and $\lambda_i-\lambda_{i-1}<\varepsilon$, $\lambda_{i-1}\leqq\lambda'_{i-1}\leqq\lambda_i$ $(i=1, 2, \ldots, k)$. Then, if we put $E_0=0$, we have $-\varepsilon I\leqq XE_\mu-\sum\limits_{i=1}^{k}\lambda'_{i-1}(E_{\lambda_i}-E_{\lambda_{i-1}})<\varepsilon I$, and thus we may write $X\cdot E_\mu=\int_0^\mu\lambda dE_\lambda$. Therefore, by $\sup\limits_{\mu>0}E_\mu=I$, we obtain the spectral representation $X=\int_0^\infty\lambda dE_\lambda$.

Proof. $XE_\mu-\sum\limits_i\lambda'_{i-1}(E_{\lambda_i}-E_{\lambda_{i-1}})=\sum\limits_i(X-\lambda'_{i-1}I)\,(E_{\lambda_i}-E_{\lambda_{i-1}})$, and moreover, by (3–7), we obtain $(\lambda_{i-1}-\lambda'_{i-1})\,(E_{\lambda_i}-E_{\lambda_{i-1}})\leqq(X-\lambda'_{i-1}I)\,(E_{\lambda_i}-E_{\lambda_{i-1}})\leqq(\lambda_i-\lambda'_{i-1})\,(E_{\lambda_i}-E_{\lambda_{i-1}})$.

§ 4. *Spectral theorem for general elements.* Let X be not necessarily positive. By (3–10) we have the spectral representation of $(X+nI)^+$, $n=1, 2, \ldots$: $(X+nI)^+=\int_0^\infty\lambda dE_\lambda(n)$, $E_\lambda(n)=I-\inf\limits_{k\geq 1}\left(\inf\limits_{m\geq 1}\left((X+nI)^+/\lambda\right)^m\right)^{\frac{1}{2k}}$. Hence, for any $\lambda>\mu>0$, we have $\mu\big(E_\lambda(n)-E_\mu(n)\big)\leqq\big(E_\lambda(n)-E_\mu(n)\big)\,(X+nI)^+\leqq\lambda\big(E_\lambda(n)-E_\mu(n)\big)$. Since $(X+nI)^+\cdot(X+nI)^-=0$ by (A–5), we have $(X+nI)^-\cdot\big(E_\lambda(n)-E_\mu(n)\big)=0$. Hence $(\mu-n)\big(E_\lambda(n)-E_\mu(n)\big)\leqq X\big(E_\lambda(n)-E_\mu(n)\big)\leqq(\lambda-n)\,\big(E_\lambda(n)-E_\mu(n)\big)$. Putting $E_\lambda(n)=E_{\lambda-n,n}$, we thus have $\mu(E_{\lambda,n}-E_{\mu,n})\leqq X(E_{\lambda,n}-E_{\mu,n})\leqq\lambda(E_{\lambda,n}-E_{\mu,n})$, $\lambda>\mu>-n$. Hence, if we put

$$E_\lambda=\sup_{n\geq 1} E_{\lambda,n}=\sup_{n>1} E_{\lambda+n}(n),$$

we obtain $E_\lambda^2=E_\lambda$, $E_\lambda \geqq E_\mu (\lambda>\mu>-\infty)$, $\mu(I-E_\mu)\leqq X(I-E_\mu)$ and $XE_\lambda \leqq \lambda E_\lambda$. Therefore, as in (3–10), we have the spectral representation

$$\begin{cases} X=\int_{-\infty}^{\infty}\lambda dE_\lambda,\ E_\lambda=E_\lambda^2=E_{\lambda-0},\ E_{-\infty}=\text{g.l.b.}_{\lambda}\ E_\lambda=0,\ E_\infty=\text{l.u.b.}_{\lambda}\ E_\lambda=I, \\ E_\lambda \geqq E_\mu\ (\lambda\geqq\mu). \end{cases}$$

It is easy to see that the spectral system $\{E_\lambda\}$ is uniquely determined by the above properties.

§ 5. *Concrete representation of the idempotents as point sets.* The set $\mathfrak{J}$ of the "idempotents" $E(E^2=E)$ of $\mathfrak{R}$ satisfies the following three conditions.

(5–1) $A, B\in\mathfrak{J}$ implies $A\vee B=A+B-AB\in\mathfrak{J}$, $A\wedge B=AB\in\mathfrak{J}$.

(5–2) $0, I\in\mathfrak{J}$, and $A\in\mathfrak{J}$ implies $I-A\in\mathfrak{J}$.

(5–3) If $A, B\in\mathfrak{J}$ are different, then either $B\leqq A$ or $A\leqq B$. Let $B\leqq A$, then there exists $C\in\mathfrak{J}$ such that $B\wedge C\neq 0$, $A\wedge C=0$. (Take, for example, $C=B\wedge(I-A)$.)

$\mathfrak{J}$ is thus a "complemented, distributive lattice" by the "join" $\vee$ and the "meet" $\wedge$, satisfying the "disjunction property" (5–3). Hence, by G. Birkhoff-Stone-Wallman's theory of Boolean ring, there exists a totally disconnected, bicompact T_1-space $\tilde{\mathfrak{J}}$ and a closed, open base $\{\tilde{A}\}$ of $\tilde{\mathfrak{J}}$ with the properties:

By a suitable correspondence $\mathfrak{J}\ni A\leftrightarrow\tilde{A}\in\tilde{\mathfrak{J}}$, $\mathfrak{J}$ is lattice-isomorphic to $\{\tilde{A}\}$, i.e. $\widetilde{(A\vee B)}=\tilde{A}\dot{+}\tilde{B}$, $\widetilde{(A\wedge B)}=\tilde{A}\cdot\tilde{B}$.—Here $\dot{+}$ and $\cdot$ denote the set-theoretic sum and product.

For the proof of this fact see, for example, H. Wallman's paper.[1)]

Moreover, by (A–6), $\mathfrak{J}$ is countably additive and countably multiplicative as lattice. This fact implies the following property:

Let $A_1<A_2<\cdots$ and $\bigvee_{i=1}^{\infty} A_i=A$. Then surely $\tilde{A}\geqq\sum_{i=1}^{\infty}\tilde{A}_i$. The closed set $(\tilde{A}-\sum_{i=1}^{\infty}\tilde{A}_i)$ does not contain any open set, i.e. $(\tilde{A}-\sum_{i=1}^{\infty}\tilde{A}_i)$ is "non-dense."

Proof. If otherwise, $(\tilde{A}-\sum_{i=1}^{\infty}\tilde{A}_i)$ would contain a certain $\tilde{B}$ with $B\neq 0$. Then, by the isomorphism $C\leftrightarrow\tilde{C}$, we must have $B\wedge A=B$ and $B\wedge A_i=0$ $(i=1, 2, \dots)$. This contradicts to $A=\bigvee_{i=1}^{\infty} A_i$.

Therefore the closed and open base $\{\tilde{A}\}$ of the topological space $\tilde{\mathfrak{J}}$ is countably additive and countably multiplicative, if we neglect the set of "first category," i.e. enumerable sum of non-dense sets. Hence, $\{\tilde{A}\}$ constitutes a "Borel field" except the set of first category. Let a real-valued function $f(t)$ on $\tilde{\mathfrak{J}}$ satisfies i) for any $\lambda>\mu$, the set $E_t\big(\lambda>f(t)>\mu\big)$ coincides with a certain $\tilde{A}$ except the set of first cate-

1) Ann. Math., **39** (1938), 112–126.

gory, and ii) the set $\underset{t}{E}(|f(t)|=\infty)$ is of first category, then we may call such function $f(t)$ as "Borel-measurable."

§ 6. *Concrete representation of* $\mathfrak{R}$ *as function ring.* By the spectral theorem we have, for all $x \in \mathfrak{R}$, $x=\int_{-\infty}^{\infty} \lambda dE_\lambda$. Let $0<\lambda_i-\lambda_{i-1}<1/n$ $(i=0 \pm 1, \pm 2, \dots)$, $\lim_{i\to-\infty} \lambda_i=-\infty$, $\lim_{i\to\infty} \lambda_i=\infty$ and consider the step functions $f_X^{(n)}(t)$ on $\tilde{\mathfrak{J}}$: $f_X^{(n)}(t)=\lambda_i$ for $t \in (\widetilde{E_{\lambda_i}-E_{\lambda_{i-1}}})$. Surely $f_X(t)=\lim_{n\to\infty} f_X^{(n)}(t)$ is Borel-measurable. It is easy to see that, by the correspondence $X \to f_X(t)$, $\mathfrak{R}$ is ring-isomorphically and lattice-isomorphically represented upon $\{f_X(t)\}$.—Here sums, products in $\{f_X(t)\}$ are ordinary functional sums and products, and $f_X(t)$ is called positive if the function $f_X(t)$ is positive except the t-set of first category.

However, to assure that any Borel-measurable function $f(t)$ is the image of some $X \in \mathfrak{R}$, it is necessary and sufficient to assume the following axiom.

(A-8) For any sequence $\{X_i\}$ with $\inf(|X_i|, |X_j|)=0$ $(i \neq j)$, we have

$$\sup_{n\geq 1} \inf_{m\leq n} \sum_{i=1}^{m} X_i=\inf_{n\geq 1} \sup_{m\leq n} \sum_{i=1}^{m} X_i \in R .$$

Proof. We have only to approximate $f(t)$ by finitely-valued Borel-measurable step functions and to put X=the limit (which surely exists by (A-8)) of the inverse images in $\mathfrak{R}$ of these step functions.

Remark 1. The axiom (A-8) is redundant, if we assume the "boundedness axiom" as stated in the Remark 2 after (A-7). For, in this case, we deal only with "bounded" Borel-measurable functions. Thus, in the "bounded case," $\mathfrak{R}$ constitutes a characterisation of the function ring of "bounded" Borel-measurable functions. Such characterisation is also announced by M. H. Stone, loc. cit. However, stone's method is different from ours.

Remark 2. In (3-3) we may take $E=\inf(X_1, I)=\inf(I, X, X^2, X^3, \dots)$. Thus, in (3-5), we may put

$$\text{(3-5)} \qquad E_\lambda=I-\inf\left(I, X/\lambda, (X/\lambda)^2, (X/\lambda)^3, \dots\right).$$

The proof is easy. Accordingly, the axioms (A-2) and (A-3) may be replaced by (2-1). (3-5)′ together with (3-10) give a lattice-theoretic interpretation of Stone-Lengyel's proof of the spectral theorem (Ann. Math. **37** (1936), 853-864). To this point I hope to discuss in another occasion.