79. *Concircular Geometry II. Integrability Conditions of* $\rho_{\mu\nu}=\phi g_{\mu\nu}$.

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In a previous paper entitled Concircular geometry I,[1] we have considered, in a Riemannian space, curves defined by

$$(0.1)\qquad \frac{\delta^3 u^\lambda}{\delta s^3}+\frac{\delta u^\lambda}{\delta s}g_{\mu\nu}\frac{\delta^2 u^\mu}{\delta s^2}\frac{\delta^2 u^\nu}{\delta s^2}=0\,, \qquad (\lambda, \mu, \nu, \ldots=1, 2, 3, \ldots, n)\,,$$

which may be regarded as a generalization of circles in ordinary euclidean space, and we have called them *geodesic circles*. If a conformal transformation

$$(0.2)\qquad \bar{g}_{\mu\nu}=\rho^2 g_{\mu\nu}$$

of the fundamental metric tensor $g_{\mu\nu}$ transforms any geodesic circle into a geodesic circle, then the functiou ρ must satisfy the following partial differential equations

$$(0.3)\qquad \rho_{\mu\nu}\equiv\rho_{\mu,\nu}-\rho_\lambda\{{}^{\lambda}_{\mu\nu}\}-\rho_\mu\rho_\nu+\frac{1}{2}g^{\alpha\beta}\rho_\alpha\rho_\beta g_{\mu\nu}=\phi g_{\mu\nu}\,, \qquad \left(\rho_\mu=\frac{\partial \log \rho}{\partial u^\mu}\right).$$

We have called such a conformal transformation a *concircular transformation.*

In the present Note, we shall consider the integrability conditions[2] of the partial differential equations (0.3).

§ 1. The function ρ satisfying the equations

$$(1.1)\qquad \rho_{\mu;\nu}-\rho_\mu\rho_\nu+\frac{1}{2}g^{\alpha\beta}\rho_\alpha\rho_\beta g_{\mu\nu}=\phi g_{\mu\nu}\,,$$

where the semi-colon denotes the covariant derivative, we have

$$(1.2)\qquad \rho_{\mu;\nu}-\rho_\mu\rho_\nu=\psi g_{\mu\nu}\,,$$

where

$$(1.3)\qquad \psi=\phi-\frac{1}{2}g^{\alpha\beta}\rho_\alpha\rho_\beta\,.$$

Consequently, putting

$$(1.4)\qquad \rho^\lambda=g^{\lambda\mu}\rho_\mu\,,$$

we obtain, from (1.2),

$$(1.5)\qquad \rho^\lambda{}_{;\nu}\rho^\nu=\rho^\lambda(\psi+\rho_\alpha\rho^\alpha)\,.$$

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1) K. Yano, Concircular geometry I. Concircular transformations. Proc. **16** (1940), 195–200.

2) This problem was also studied by A. Fialkow, Conformal geodesics, Trans. Amer. Math. Soc. **45** (1939), 443–473.

The equations (1.5) show that a curve $u^\lambda(t)$ whose tangential direction coincides with that of the vector ρ^λ, is a geodesic. We shall call such a curve *ρ-curve.*

Thus we have the

Theorem I. If the conformal transformation (0.2) *is a concircular one, the ρ-curves are geodesics.*

Let us now consider a hypersurface defined by

$$\rho(u^\lambda)=\text{const.} \tag{1.6}$$

This hypersurface is also represented by the equations of the forme

$$u^\lambda=u^\lambda(u^i) \qquad (i, j, k, \ldots=\dot{1}, \dot{2}, \ldots, \dot{n}-\dot{1}) \tag{1.7}$$

the u^i's being parameters on the hypersurface. If we substitute the u^λ's given by (1.7) into (1.6), (1.6) must be reduced to an identity.

Consequently, differentiating (1.6) logarithmically with respect to u^j, we have

$$\rho_\mu B_j^{\cdot\mu}=0\,, \tag{1.8}$$

where

$$\rho_\mu=\frac{\partial \log \rho}{\partial u^\mu} \quad \text{and} \quad B_j^{\cdot\mu}=\frac{\partial u^\mu}{\partial u^j}\,.\text{[1)]} \tag{1.9}$$

Differentiating (1.8) once more covariantly, we obtain

$$\rho_{\mu;\,\nu}B_j^{\cdot\mu}B_k^{\cdot\nu}+\rho_\mu H_{jk}^{\cdot\cdot\mu}=0 \tag{1.10}$$

where

$$H_{jk}^{\cdot\cdot\mu}=\frac{\partial^2 u^\mu}{\partial u^j \partial u^k}+B_j^{\cdot\alpha}B_k^{\cdot\beta}\{{}^{\,\mu}_{\alpha\beta}\}-B_i^{\cdot\mu}\{{}^{\,i}_{jk}\}\,. \tag{1.11}$$

Substituting

$$\rho_{\mu;\,\nu}=\psi g_{\mu\nu}+\rho_\mu\rho_\nu\,,$$

into (1.10) and taking account of (1.8), we have

$$\psi g_{jk}+\rho_\mu H_{jk}^{\cdot\cdot\mu}=0\,. \tag{1.12}$$

Contracting g^{jk}, we have, from (1.12),

$$\psi=\frac{-1}{n-1}\rho_\mu g^{ab}H_{ab}^{\cdot\cdot\mu}\,,$$

then (1.12) becomes

$$\rho_\mu M_{jk}^{\cdot\cdot\mu}=0\,, \tag{1.13}$$

where

$$M_{jk}^{\cdot\cdot\mu}=H_{jk}^{\cdot\cdot\mu}-\frac{1}{n-1}g^{ab}H_{ab}^{\cdot\cdot\mu}g_{jk}\,. \tag{1.14}$$

But we know, on the other hand, that the $M_{jk}^{\cdot\cdot\mu}$ are vectors normal to the hypersurface with respect to the index μ, and ρ_μ is a vector also normal to the hypersurface, so that we conclude from (1.13) that

1) For the notations, see K. Yano, Sur les équations de Gauss dans la géométrie conforme des espaces de Riemann, Proc. **15** (1939), 247–252 and Sur les équations de Codazzi dans la géométrie conforme des espaces de Riemann, Proc. **15** (1939), 340–344.

$$(1.15) \qquad M_{jk}^{\cdot\cdot\mu}=0$$

ρ_μ not being identically zero. Thus we have the

Theorem II. If the conformal transformation (0.2) *is a concircular one, the hypersurfaces ρ=const. are totally umbilical.*

We shall call these hyperfaces *ρ-hypersurfaces.*

We shall now differentiate

$$(1.16) \qquad \rho^\lambda{}_{;\,\mu}=\psi\delta^\lambda_\mu+\rho^\lambda\rho_\mu$$

covariantly, then we obtain

$$\rho^\lambda{}_{;\,\mu;\,\nu}=\psi_{;\,\nu}\delta^\lambda_\mu+\rho^\lambda_{;\,\nu}\rho_\mu+\rho^\lambda\rho_{\mu;\,\nu}\,.$$

Substituting (1.2) and (1.16) in the above equations, we find

$$\rho^\lambda{}_{;\,\mu;\,\nu}=\psi_{;\,\nu}\delta^\lambda_\mu+(\psi\delta^\lambda_\nu+\rho^\lambda\rho_\nu)\rho_\mu+\rho^\lambda(\psi g_{\mu\nu}+\rho_\mu\rho_\nu)\,.$$

Commutating μ and ν and subtracting, we obtain

$$(1.17) \qquad \rho^\alpha R^{\lambda}_{\cdot\alpha\mu\nu}=\delta^\lambda_\mu\psi_{;\,\nu}-\delta^\lambda_\nu\psi_{;\,\mu}+\psi\rho_\mu\delta^\lambda_\nu-\psi\rho_\nu\delta^\lambda_\mu\,.$$

Multiplying (1.17) by ρ_λ and summing up for the index λ, we have

$$\rho_\mu\psi_{;\,\nu}-\rho_\nu\psi_{;\,\mu}=0\,,$$

because of the identity

$$\rho_\lambda\rho^\alpha R^{\lambda}_{\cdot\alpha\mu\nu}=\rho^\lambda\rho^\alpha R_{\lambda\alpha\mu\nu}=0\,.$$

Multiplying these equations by ρ^μ and summing up for μ, we obtain

$$(1.18) \qquad \psi_{;\,\nu}=\frac{\psi_{;\,\lambda}\rho^\lambda}{\rho_\alpha\rho^\alpha}\rho_\nu\,.$$

We put next $\lambda=\nu$ in (1.17) and sum up, then we find

$$\rho^\alpha R_{\alpha\mu}=(n-1)(\psi\cdot\rho_\mu-\psi_{;\,\mu})\,.$$

Substituting (1.18) in these equations, we obtain finally

$$(1.19) \qquad R^{\lambda}_{\cdot\alpha}\rho^\alpha=(n-1)\left(\psi-\frac{\psi_{;\,\mu}\rho^\mu}{\rho_\alpha\rho^\alpha}\right)\rho^\lambda\,,$$

which shows that the vector ρ^λ is in a Ricci-direction, thus we have the

Theorem III. If the conformal transformation (0.2) *is a concircular one, the ρ-curves are Ricci-curves.*

§ 2. In the preceding paragraph, we have seen that there exists, in our Riemannian space, a family of ∞^1 totally umbilical hypersurfaces the orthogonal trajectories of which are geodesic Ricci-curves. In the present paragraph, we shall show that if these geometrical conditions are satisfied, our Riemannian space admits at least a solution of the partial differential equations (1.2).

Let us choose a coordinate system in which u^n=const. defines the family of totally umbilical hypersurfaces and u^i=const. $(i, j, k, \dots = 1, 2, 3, \dots, n-1)$ define the orthogonal trajectories of the totally umbilical hypersurfaces.

Then we have at first

(2.1) $$g_{ni}=g_{in}=0\,.$$

The curves defined by

$$u^1=\text{const.}\quad u^2=\text{const.}\ \cdots\quad u^{n-1}=\text{const.}\quad u^n=u^n\,,$$

being geodesics, we have

$$\frac{du^\lambda}{du^n}=\delta^\lambda_n$$

$$\frac{d}{du^n}\frac{du^\lambda}{du^n}+\{{}^{\lambda}_{\mu\nu}\}\frac{du^\mu}{du^n}\frac{du^\nu}{du^n}=a\cdot\frac{du^\lambda}{du^n}$$

hence

$$\{{}^{\lambda}_{nn}\}=a\cdot\delta^\lambda_n$$

from which we obtain

(2.2) $$\{{}^{i}_{nn}\}=0 \quad\text{or}\quad \frac{1}{2}g^{ia}\Big(\frac{\partial g_{an}}{\partial u^n}+\frac{\partial g_{an}}{\partial u^n}-\frac{\partial g_{nn}}{\partial u^a}\Big)=0\,.$$

On account of the equations $g^{in}=0$ and $g_{jn}=0$, we have, from (2.2),

$$\frac{\partial g_{nn}}{\partial u^j}=0\,.$$

These equations show that the function g_{nn} must be of the form

(2.3) $$g_{nn}=g_{nn}(u^n)\,.$$

The hypersurfaces defined by $u^1=u^1, u^2=u^2, \dots, u^{n-1}=u^{n-1}, u^n=\text{const.}$ being totally umbilical, we have

(2.4) $$B_i^{\cdot\lambda}=\frac{\partial u^\lambda}{\partial u^i}=\delta^\lambda_i$$

(2.5) $$H_{jk}^{\cdot\cdot\lambda}=\frac{\partial^2 u^\lambda}{\partial u^j\partial u^k}+\frac{\partial u^\mu}{\partial u^j}\frac{\partial u^\nu}{\partial u^k}\{{}^{\lambda}_{\mu\nu}\}-\frac{\partial u^\lambda}{\partial u^i}\{{}^{i}_{jk}\}=g_{jk}H^\lambda\,.$$

From (2.4) and (2.5), we find

(2.6) $$\{{}^{\lambda}_{jk}\}-\delta^\lambda_i\{{}^{i}_{jk}\}=g_{jk}H^\lambda\,.$$

Putting $\lambda=n$ in this equation, we find

(2.7) $$\{{}^{n}_{jk}\}=g_{jk}H^n \quad\text{or}\quad \frac{1}{2}g^{na}\Big(\frac{\partial g_{aj}}{\partial u^k}+\frac{\partial g_{ak}}{\partial u^j}-\frac{\partial g_{jk}}{\partial u^a}\Big)=g_{jk}H^n\,.$$

On account of the relations $g^{ni}=0$ and $g_{nj}=0$, we have, from (2.7),

(2.8) $$-\frac{1}{2}g^{nn}\frac{\partial g_{jk}}{\partial u^n}=g_{jk}H^n\,.$$

The equations (2.8) show that the functions g_{jk} have the form[1)]

(2.9) $$g_{jk}=\sigma(u^\lambda)f_{jk}(u^i)\,.$$

1) Cf. K. Yano: Conformally separable quadratic differential forms. Proc. **16** (1940), 83-86.

The equations (2.1), (2.3) and (2.9) show that the fundamental quadratic differential form $ds^2=g_{\mu\nu}du^\mu du^\nu$ must be written as

(2.10) $$ds^2=\sigma(u^\lambda)f_{jk}(u^i)du^j du^k+g_{nn}(u^n)du^n du^n\,.$$

We know moreover that the curves defined by $'u^1=\text{const.}, \ldots, u^{n-1}=\text{const.}\ u^n=u^n$ are Ricci-curves. We have then the equations of the form

$$R^{\lambda}_{\cdot\mu}\frac{du^\mu}{du^n}=\alpha\cdot\frac{du^\lambda}{du^n}\,,$$

from which we find

$$R^{i}_{\cdot n}=g^{i\lambda}R_{\lambda n}=0\,,$$

or

(2.11) $$R_{in}=0\,.$$

To calculate the components R_{in} of the Ricci-tensor, we shall, taking account of (2.10), write down the values of the Christoffel symbols as follows:

(2.12) $$\left\{\begin{aligned} &\left\{{i \atop jk}\right\}=\frac{1}{2}f^{ia}\left(\frac{\partial f_{aj}}{\partial u^k}+\frac{\partial f_{ak}}{\partial u^j}-\frac{\partial f_{jk}}{\partial u^a}\right)+\frac{1}{2}\delta^i_j\frac{\partial \log\sigma}{\partial u^k}+\frac{1}{2}\delta^i_k\frac{\partial\log\sigma}{\partial u^j}\\ &\qquad\qquad -\frac{1}{2}f^{ia}\frac{\partial\log\sigma}{\partial u^a}f_{jk}\,,\\ &\left\{{n \atop jk}\right\}=-\frac{1}{2}g^{nn}\frac{\partial\log\sigma}{\partial u^n}g_{jk}\,,\qquad \left\{{i \atop jn}\right\}=\left\{{i \atop nj}\right\}=\frac{1}{2}\frac{\partial\log\sigma}{\partial u^n}\delta^i_j\,,\\ &\left\{{i \atop nn}\right\}=0\,,\qquad \left\{{n \atop jn}\right\}=\left\{{n \atop nj}\right\}=0\,,\qquad \left\{{n \atop nn}\right\}=\frac{1}{2}g^{nn}\frac{\partial g_{nn}}{\partial u^n}\,,\end{aligned}\right.$$

where

$$f^{ia}f_{aj}=\delta^i_j \quad\text{and}\quad g^{nn}=\frac{1}{g_{nn}}\,.$$

We have then

$$\begin{aligned} R_{in}=R^{\lambda}_{\cdot in\lambda}=R^{j}_{\cdot inj}&=\frac{\partial\left\{{j \atop in}\right\}}{\partial u^j}-\frac{\partial\left\{{j \atop ij}\right\}}{\partial u^n}+\left\{{\lambda \atop in}\right\}\left\{{j \atop \lambda j}\right\}-\left\{{\lambda \atop ij}\right\}\left\{{j \atop \lambda n}\right\}\\ &=\frac{1}{2}\frac{\partial^2\log\sigma}{\partial u^i\partial u^n}-\frac{n-1}{2}\frac{\partial^2\log\sigma}{\partial u^n\partial u^i}+\frac{1}{2}\frac{\partial\log\sigma}{\partial u^n}\left\{{j \atop ij}\right\}-\frac{1}{2}\frac{\partial\log\sigma}{\partial u^n}\left\{{j \atop ij}\right\}\\ &=\frac{2-n}{2}\frac{\partial^2\log\sigma}{\partial u^i\partial u^n}\,.\end{aligned}$$

The above equations and (2.11) show that

(2.13) $$\sigma(u^\lambda)=g(u^i)h(u^n)\,,$$

then the fundamental form must be of the form

(2.14) $$ds^2=\sigma(u^n)f_{jk}(u^i)du^j du^k+g_{nn}(u^n)du^n du^n\,.$$[1)]

The fundamental form being reduced to the form (2.14), we shall

1) A. Fialkow: Conformal geodesics, loc. cit. p. 471.

now consider a function $\rho(u^n)$ defined by

$$(2.15)\qquad \log\rho=-\log\int\sqrt{\sigma(u^n)g_{nn}(u^n)}\,du^n\,,^{1)}$$

then we have

$$\rho_\mu=-\frac{\sqrt{\sigma g_{nn}}}{\int\sqrt{\sigma g_{nn}}\,du^n}\delta^n_\mu\,,$$

$$\rho_{\mu;\,\nu}=\frac{\sigma g_{nn}}{\left(\int\sqrt{\sigma g_{nn}}\,du^n\right)^2}\delta^n_\mu\delta^n_\nu-\frac{\frac{d\sigma}{du^n}g_{nn}+\sigma\frac{dg_{nn}}{du^n}}{2\sqrt{\sigma g_{nn}}\int\sqrt{\sigma g_{nn}}\,du^n}\delta^n_\mu\delta^n_\nu+\frac{\sqrt{\sigma g_{nn}}}{\int\sqrt{\sigma g_{nn}}\,du^n}\delta^n_\lambda\left\{{\lambda \atop \mu\nu}\right\},$$

$$\rho_{\mu;\,\nu}-\rho_\mu\rho_\nu=-\frac{\sqrt{\sigma g_{nn}}}{\int\sqrt{\sigma g_{nn}}\,du^n}\left[\left(\frac{1}{2}\frac{d\log\sigma}{du^n}+\frac{1}{2}\frac{d\log g_{nn}}{du^n}\right)\delta^n_\mu\delta^n_\nu-\left\{{n \atop \mu\nu}\right\}\right].$$

Putting $\mu=j,\ \nu=k$; $\mu=j,\ \nu=n$; $\mu=n,\ \nu=n$ respectively and taking account of (2.12), we have

$$\rho_{j;\,k}-\rho_j\rho_k=-\frac{\sqrt{\sigma g_{nn}}}{2\int\sqrt{\sigma g_{nn}}\,du^n}g^{nn}\frac{d\log\sigma}{du^n}g_{jk}\,,$$

$$\rho_{j;\,n}-\rho_j\rho_n=0\,,$$

$$\rho_{n;\,n}-\rho_n\rho_n=-\frac{\sqrt{\sigma g_{nn}}}{\int\sqrt{\sigma g_{nn}}\,du^n}\left[\frac{1}{2}\frac{d\log\sigma}{du^n}+\frac{1}{2}\frac{d\log g_{nn}}{du^n}-\frac{1}{2}g^{nn}\frac{dg_{nn}}{du^n}\right]$$

$$=-\frac{\sqrt{\sigma g_{nn}}}{2\int\sqrt{\sigma g_{nn}}\,du^n}g^{nn}\frac{d\log\sigma}{du^n}g_{nn}\,.$$

These equations show that

$$\rho_{\mu;\,\nu}-\rho_\mu\rho_\nu=-\frac{\sqrt{\sigma g_{nn}}}{2\int\sqrt{\sigma g_{nn}}\,du^n}g^{nn}\frac{d\log\sigma}{du^n}g_{\mu\nu}\,.$$

Thus we have proved that the function defined by (2.15) is a solution of the partial differential equations (1.2). Thus we have the

Theorem IV. A necessary and sufficient condition that a Riemannian space admit a solution of the partial differential equations (1.2) *is that the Riemannian space contain a family of totally umbilical hypersurfaces whose orthogonal trajectories are geodesic Ricci-curves.*

Remark. The differential equations of a generalized circle defined by the present author[2] are given by

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1) This function was suggested to the author by Prof. A. Kawaguchi.

2) K. Yano, Sur les circonférences généralisées dans les espaces à connexion conforme, Proc. **14** (1938), 329–332.

$$\frac{\delta^3 u^\lambda}{\delta s^3}+\frac{\delta u^\lambda}{\delta s}g_{\mu\nu}\frac{\delta^2 u^\mu}{\delta s^2}\frac{\delta^2 u^\nu}{\delta s^2}+\frac{1}{n-2}\frac{\delta u^\lambda}{\delta s}R_{\mu\nu}\frac{\delta u^\mu}{\delta s}\frac{\delta u^\nu}{\delta s}-\frac{1}{n-2}R^{\lambda}_{.\nu}\frac{\delta u^\nu}{\delta s}=0.$$

These equations show that if a curve belongs to any two of the following classes of curves, it belongs also to the third:

(I) geodesic circles (II) Ricci-curves (III) generalized circles.

The ρ-curves belong to the first and the second class, then they belong also to the third, that is to say, the ρ-curves may be regarded as generalized circles.