116. An Abstract Integral, III.

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The object of this paper is to make the integration theory free from the concept of function.

1. Let $\mathbb{L}$ be a system of elements $a, b, c, \dots, x, y, z, \dots$ and let $\alpha, \beta, \gamma, \dots$ be real numbers and $k, m, n, \dots$ be integers. We suppose that $\mathbb{L}$ satisfies the following axioms.

Axiom 1. $\mathbb{L}$ is an abelian group with real number field as operator domain. Group operation is denoted by " $+$ ".

Axiom 2. $\mathbb{L}$ is partially ordered, that is, the relation " $\leqq$ " is defined and

(2.1) $a \leqq a$,

(2.2) $a \leqq b$ and $b \leqq c$ imply $a \leqq c$.

Axiom 3. $\mathbb{L}$ is a lattice, that is, for every a and every b in $\mathbb{L}$, there exist the join $a \cup b$ and the meet $a \cap b$ such that

(3.1) $a \leqq a \cup b$, $b \leqq a \cup b$, and $a \leqq c$, $b \leqq c$ imply $a \cup b \leqq c$,

(3.2) $a \geqq a \cap b$, $b \geqq a \cap b$, and $a \geqq d$, $b \geqq d$ imply $a \cap b \geqq d$.

Axiom 3′. $\mathbb{L}$ is a "restricted" σ-lattice, that is, for any "bounded"[1] sequence $\{x_n\}$, there exist the elements $\bigvee_{n=1}^{\infty} x_n$ and $\bigwedge_{n=1}^{\infty} x_n$ such that

(3′.1) $x_m \leqq \bigvee_{n=1}^{\infty} x_n$ $(m=1, 2, \dots)$ and $x_n \leqq c'$ $(n=1, 2, \dots)$ imply $\bigvee_{n=1}^{\infty} x_n \leqq c'$,

(3′.2) $x_m \geqq \bigwedge_{n=1}^{\infty} x_n$ $(m=1, 2, \dots)$ and $x_n \geqq d'$ $(n=1, 2, \dots)$ imply $\bigwedge_{n=1}^{\infty} x_n \geqq d'$.

Axiom 4. Between partially ordering and group operation there hold the relations:

(4.1) $a > 0$ implies $-a < 0$,

(4.2) $a > b$ implies $a+c > b+c$,

(4.3) $\alpha > 0$ and $a > 0$ imply $\alpha a > 0$.

We need further some definitions.

Definition 1. $x^+ = x \cup 0$, $x^- = x \cap 0$ and $|x| = x^+ - x^-$.

Definition 2. $\overline{\lim_{n\to\infty}} x_n = \bigwedge_{n=1}^{\infty} (\bigvee_{m=n}^{\infty} x_m)$, $\underline{\lim_{n\to\infty}} x_n = \bigvee_{n=1}^{\infty} (\bigwedge_{m=n}^{\infty} x_m)$, provided that $\{x_n\}$ is bounded. If they coincide, then we denote it by $\lim_{n\to\infty} x_n$.

2. We will now define the abstract Riemann and Lebesgue integral of element of $\mathbb{L}$. We will begin by the

1) Let $S < \mathbb{L}$. If there are u and l in $\mathbb{L}$ such that $l \leqq s \leqq u$ for all s in S, then S is called bounded.

Definition 3. If S is a subset of $\mathbb{L}$ and for every a and every b in S $\alpha a+\beta b$ belongs to S, then S is called linear.

Definition 4. If S is linear and $x\varphi$ is a functional such that $x\geqq 0$ implies $x\varphi\geqq 0$, then φ is called positive. If $(\alpha a+\beta b)\varphi=\alpha(a\varphi)+\beta(b\varphi)$, then φ is called linear.

Definition 5[1]. $f=f_1$ is called an (abstract) Riemann integral and $R=R_1$ the class of Riemann integrable elements, provided that

[5.1] $R\subset\mathbb{L}$ and R is linear,

[5.2] f is a functional defined for all elements in R and non-negative and linear.

[5.3] $x\in R$ implies $|x|\in R$,

[5.4] if $\{x_n\}\subset R$, $\lim_{n\to\infty} x_n=0$ and there is a $y\in R$ such that $|x_n|\leqq y$ $(n=1,2,\dots)$, then $\lim_{n\to\infty} f(x_n)=0$.

Definition 6[1]. $F=F_1$ is called an (abstract) Lebesgue integral and $L=L_1$ is the class of lebesgue integrable (or shortly L-integrable) elements provided that

[6.1] $L\subset\mathbb{L}$ and L is linear,

[6.2] F is a functional defined for all elements in L and non-negative and linear,

[6.3] $z\in L$ implies $|z|\in L$,

[6.4] if $z\in L$, $zF=0$ and $|y|\leqq z$, then $yF=0$,

[6.5] if $\{z_n\}\subset L$, $\lim z_n=z$ and there exists a $y\in L$ such that $|z_n|\leqq y$ $(n=1,2,\dots)$ then $z\in L$ and $\lim_{n\to\infty} z_nF=zF$,

[6.6] if $\{z_n\}\subset L$, $z_n\leqq z_{n+1}$ $(n=1,2,\dots)$, $\lim_{n\to\infty} z_n=z$ and $\lim_{n\to\infty} z_nF$ is finite, then $z\in L$ and $\lim_{n\to\infty} z_nF=zF$.

3. From the definitions above stated we can prove

Theorem 1[2]. If f is a Riemann integral and R is the class of R-integrable elements, then there are Lebesgue integral F and the class of L-integrable elements L such that

{1.1} $R\subseteqq L$,

{1.2} $xf=xF$ for all x in R.

Proof. We define L' as the set of z such that there are sequences $\{x_n\}$ and $\{y_n\}$ in R such as

$$\varliminf_{n\to\infty} x_n\geqq z\geqq \varlimsup_{n\to\infty} y_n\,.$$

Then $L'\geqq R$. By $z\bar{F}$ we mean the greatest lower bound of $\lim_{n\to\infty} x_nf$ where $\{x_n\}\subset R$, $\varliminf_{n\to\infty} x_n\geqq z$ and there is a y in R such that $x_n\geqq y$ $(n=1,2,\dots)$. We put $z\underline{F}=-(-z)\bar{F}$. Let L be the set of z such that $z\bar{F}=z\underline{F}$ and zF is defined by the common value. F and L thus defind, satisfy the required conditions and axioms.

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1) This definition is essentially due to Daniell and Banach.

2) This theorem is essentially due to Daniell and Banach.

4. Let us now introduce the notion of product of elements. For this purpose we replace Axiom 1 by the following axiom:

Axiom 1′. $\mathbb{L}$ is a commutative ring with real number field as operator domain. Ring operations are denoted by "+" and ".".

Axiom 4 is added by

(4.4) $a>0$ and $b>0$ imply $ab>0$,

(4.5) If unit element 1 exist, then $1>0$.

In such a lattice $\mathbb{L}$, we define the second Riemann integral f_2 and the class of R-integrable elements R_2 such that the conditions [5.1]-[5.4] hold good and further

[5.1′] unit 1 belongs to R_2 and $1f_2=1$[1].

[5.3′] $x \in R_2$ and $y \in R_2$ imply $xy \in R_2$.

The second Lebesgue integral F_2 and the class of L-integrable functions L_2 is defined such that conditions [6.1]-[6.6] hold good.

Then we have

Theorem 2. If f_2 is the second Riemann integral and R_2 is the class of R-integrable elements, then there are Lebesgue integral F_2 and the class of L-integrable elements L_2 such that

{2.1} $R_2 \subseteqq L_2$,

{2.2} $xf_2=xF_2$ for all x in R_2,

{2.3′} $x \in R_2$ and $z \in L_2$ imply $xz \in L_2$.

Proof. Let L'' be the set of z such that there are sequences $\{x_n\}$ and $\{y_n\}$ in R such as

$$\varliminf_{n\to\infty} xx_n \geqq xz \geqq \overline{\lim_{n\to\infty}}\, xy_n$$

for all x in R_2. We have $L'' \geqq R_2$.

We define $(x,z)\bar{F}_2$ as the greatest lower bound of $\varliminf_{n\to\infty}(xx_n)f$ where $\{x_n\}\subset R$, $\varliminf_{n\to\infty} x_n \geqq z$ and there is a y in R such that $x_n \geqq y$ $(n=1, 2, \ldots)$. We put $(x,z)\underline{F}_2=-(x,-z)\bar{F}_2$. If $(x,z)\underline{F}_2$ and $(x,z)\bar{F}_2$ are finite and equal for all x in R, then we denote it by $(xz)F_2$ and the set of all z for which $(xz)F_2$ is defined, by L_2.

It is easy to verify that F_2 and L_2 are the required ones.

5. We will consider the third Riemann integral f_3 and R_3 such that f_3 and R_3 satisfy the conditions [5.1], [5.1′], [5.2], [5.3], [5.3′], [5.4] and

[5.5] R contains a complete orthogonal system $\{x_n\}$, that is, (1°) $\{x_n\}$ is a normalized orthogonal system, i. e.

$$(x_i x_j)f_3=\begin{cases} 0 & \text{for } i\neq j, \\ 1 & \text{for } i=j, \end{cases}$$

and (2°) $\{x_n\}$ is complete in L_2, that is, $(x_i z)F_2=0$ $(i=1,2,\ldots)$ imply $z=0$.

For a z in L_2, we put

1) This axiom is used only in § 6.

$$a_i=(x_iz)F_2 \qquad (i=1, 2, \dots)$$

which is called Fourier coefficients of z. Thus we get the formal series

$$a_ix_i=a_1x_1+a_2x_2+\cdots+a_nx_n+\cdots$$

which is called Fourier series of z. This is a representation of z in L_2, so that we write

$$z\sim\sum_{i=1}^{\infty}a_ix_i\,.$$

6. We will define the fourth Riemann integral f_4 and R_4 which are f_3 and R_3 satisfying the condition:

[5.3″] if $x\in R_4$ and $y\in R_4$, then $xy\in R_4$ and $(xy)f_4\leqq(xx)f_4(yy)f_4$.

By F_4 and L_4, $L_4^{(2)}$, we mean the Lebesgue integral and class of L-integrable elements, its subspace which satisfy [6.1]-[6.6], and

[7.1] $L_4^{(2)}\subseteqq L$,

[7.2] for every w and every z in $L_4^{(2)}$, there exists $(zw)F_4$ and

$$\big((zw)F_4\big)^2\leqq(zz)F_4\cdot(ww)F_4\,.$$

Theorem 3. If f_4 is the fourth Riemann integral and R_4 is the class of R-integrable elements, then there are Lebesgue integral F_4 and the class of L-integrable elements L_4 and its subspace $L_4^{(2)}$ such that

{3.1} $R_4\subseteqq L_4^{(2)}\subseteqq L_4$,

{3.2} $xf_4=xF_4$ for all x in R_4,

{3.3} for every z in R_4 and w in $L_4^{(2)}$, there exists $(wz)F_4$ and

$$\big((zw)F_4\big)^2\leqq(zz)F_4(ww)F_4\,.$$

Proof. We define L by the set of $w, z, \dots$ such that for every w and every z in L there exists $\{x_n^1\}$, $\{y_n^1\}$, $\{x_n^2\}$, $\{y_n^2\}$ in R such that

$$\underline{\lim}_{n\to\infty}x_n^1\geqq z\geqq\overline{\lim}_{n\to\infty}x_n^2\,,\qquad \underline{\lim}_{n\to\infty}y_n^1\geqq w\geqq\overline{\lim}_{n\to\infty}y_n^2$$

and

$$\underline{\lim}_{n\to\infty}x_n^1y_n^1\geqq zw\geqq\overline{\lim}_{n\to\infty}x_ny_n\,,$$

where zw need not belong to L_4^*. By $(w,z)F_4^*$ we denote the greatest lower bound of $\underline{\lim}_{n\to\infty}(x_ny_n)f$ such that $\{x_n\}$ and $\{y_n\}$ belong to R_4, $\underline{\lim}_{n\to\infty}x_ny_n\geqq wz$, $\underline{\lim}_{n\to\infty}x_n\geqq w$, $\underline{\lim}_{n\to\infty}y_n\geqq z$ and there are x' and y' such that $x_n\geqq x'$, $y_n\geqq y'$ $(n=1,2,\dots)$. We put $(w,z)\underline{F}=-(-w,z)\overline{F}=-(w,-z)\overline{F}$. Let $L_4^{(2)}$ be the set of z such that $(w,z)\overline{F}=(w,x)\underline{F}$ for all w in $L_4^{(2)}$, and the common value is denoted by $(wz)F$. And L_4 be the set of z such that $(w,x)\overline{F}=(w,z)\underline{F}$ for all w in R_4, and $(lw)F$ is denoted by wF_4, which is called the Lebesgue integral of z in L_4. Thus defined L_4, $L_4^{(2)}$ and F_4 satisfy the required conditions.

Since $L_4^{(2)}$ is contained in L_4, we can define the Fourier series of z in $L_4^{(2)}$

$$z \sim \sum_{n=1}^{\infty} a_n x_n .$$

By the assumption of $\mathbb{L}$ and additivity of F_4, we get

$$\Big((z-\sum_{n=1}^{N} a_n x_n)\,(z-\sum_{n=1}^{N} a_n x_n)\Big)F_4=(z\cdot z)F_4-\sum_{n=1}^{N} a_n^2 .$$

By $0F=0$ and [7.2] (putting $w=0$), the left hand side is $\geqq 0$. Thus we get the Bessel's inequality

$$(zz)F \geqq \sum_{n=1}^{\infty} a_n^2 .$$

7. In order to prove the Riesz-Fischer theorem we will further introduce the assumption:

[6.6′] if $\{z_n\} \subset L_4$, $0 \leqq z_n \leqq z_{n+1}$ and $\lim (z_n)F_4 < \infty$, then there exists $\lim z_n$.

This assumption includes that if $\{z_n\} \subset L_4$ and $\sum_{n=1}^{\infty} (|z_n - z_{n-1}|)F_4 < \infty$, then there exists the element $\bigvee_{n=1}^{\infty} z_n = \lim_{n\to\infty} z_n$.

We need a lemma:

Lemma. If $\{z_n\} \subset L_4^2$, then $\lim_{m,n\to\infty} \{(z_m - z_n)\,(z_m - z_n)\}F_4 = 0$ implies the existence of z in $L_4^{(2)}$, such that $\lim_{n\to\infty} \{(z_n - z)\,(z_n - z)\}\,F_4 = 0$.

Proof. Necessity of the condition is easy by the Minkowski's inequality, which is evident by [7.2]. For the proof of sufficiency, we put

$$\delta_i = \max_{m,n \geq i} \{(z_m - z_n)\,(z_m - z_n)\}\,F_4 .$$

Since $\delta_i \to 0$, there exists an increasing sequence n_k such that $\sum_{k=1}^{\infty} \delta_{n_k}$ converges. Therefore

$$(|z_{n_{k+1}} - z_{n_k}|)F_4 \leqq \delta_{n_k} \qquad (k=1, 2, \dots) .$$

By [6.6′], there exists $z = \lim_{k\to\infty} z_{n_k}$.

We have also $\{(z_m - z_{n_k})\,(z_m - z_{n_k})\}F_4 \leqq \delta_m$ for all $n_k > m$, and then [6.6′] gives us

$$\{(z_m - z_n)\,(z_m - z_n)\}F_4 \leqq \delta_m \qquad (m=1, 2, \dots)$$

which is the required.

We can now prove the Riesz-Fischer theorem:

Theorem 4. If $\sum_{n=1}^{\infty} a_n^2 < \infty$, then there exists an elements z in $L_4^{(2)}$, such that $\{a_i\}$ are Fourier coefficients of z and

$$(zz)F_4 = \sum_{i=1}^{\infty} a_i^2 , \qquad \lim_{n\to\infty} \{(z - s_n)\,(z - s_n)\}F_4 = 0 ,$$

where $s_n = a_1 x_1 + \cdots + a_n x_n$.

Proof. We have $\qquad \{(s_m - s_n)\,(s_m - s_n)\}F_4 = \sum_{i=n+1}^{m} a_i^2$

which tends to zero as $m, n \to \infty$. Lemma gives the existence of z in $L_4^{(2)}$ such that

$$\lim_{n\to\infty} \{(s_n - z)(s_n - z)\} F_4 = 0 .$$

And we have

$$a_i = (x_i s_n) F_4 = (x_i z) F_4 + \big(x_i (s_n - z)\big) F_4 ,$$

where

$$\big(x_i (s_n - z)\big) F_4 \leqq \big((s_n - z)(s_n - z)\big) F_4 \to 0 .$$

Therefore $\{a_i\}$ are the Fourier coefficients of z.
On the other hand

$$(s_{n_k} - s_{n_k}) F_4 = \sum_{i=1}^{n_k} a_i^2 \leqq \sum_{i=1}^{\infty} a_i^2 .$$

By [6.6], we have

$$(zz) F_4 \leqq \sum_{i=1}^{\infty} a_i^2 .$$

Combining this with the Bessel inequality, we get the identity

$$(zz) F_4 = \sum_{i=1}^{\infty} a_i^2 .$$