14. *Algebras with Anti-isomorphic Left and Right Ideal Lattices.*

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Let A be an algebra over a field F. A is called *Frobeniusean* if it has a unit element and if its left and right regular representations $L(a)$ and $R(a)$ $(a \in A)$ are equivalent to each other. If however A has a unit element and the totality of distinct directly indecomposable components of $L(a)$ coincides with that of $R(a)$, then we call A *quasi-Frobeniusean.*[1] In a previous paper we have shown,[2] among others: If A is quasi-Frobeniusean then the relation to annihilate each other gives a (one-one) anti-isomorphism between the left and the right ideal lattices Λ and P of A, that is,

$$\alpha) \qquad l\big(r(\mathfrak{l})\big)=\mathfrak{l}\,, \qquad r\big(l(\mathfrak{r})\big)=\mathfrak{r}$$

for every left ideal $\mathfrak{l}$ and right ideal $\mathfrak{r}$, where $l(S)$ $[r(S)]$ $(S \subseteqq A)$ denotes the set of left [right] annihilators of S in A, and conversely. Furthermore, if A is Frobeniusean then this anti-isomorphism of Λ and P is such that the corresponding left and right ideals have dual ranks with respect to F:

$$\beta) \qquad (\mathfrak{l}:F)+\big(r(\mathfrak{l}):F\big)=(A:F)\,, \qquad (\mathfrak{r}:F)+\big(l(\mathfrak{r}):F\big)=(A:F)\,,$$

and conversely.

Now, do the second parts of these theorems remain valid if we assume simply the existence of an anti-isomorphism between Λ and P, or a one such that the corresponding left and right ideals have dual ranks over F, which may not be given rise by the annihilation relation? The answer is positive in both cases, as we want to see in the present note. (As a matter of fact, the converse parts of our previous main theorems are somewhat sharper than the ones quoted above, and those sharper results are, contrary to the above milder ones, *not* covered by those of the present note.)

In the sequel we shall however deal with rings satisfying the minimum condition (for left and right ideals), rather than the special case of algebras. Let A be such a ring, and N its radical. Let further $e_{\kappa,i}$ $\big(\kappa=1, 2, \dots, k\,;\ i=1, 2, \dots, f(\kappa)\big)$ be a maximal system of mutually orthogonal primitive idempotent elements in A, such that two left ideals $Ae_{\kappa,i}$ and $Ae_{\lambda,j}$ are operator-isomorphic if and only if $\kappa=\lambda$. Then the same is the case for the right ideals generated by them. A is decomposed into direct sums

1) On Frobeniusean algebras, I., Ann. Math. **40** (1939), p. 611–33; Part II., ibid. **42** (1941), p. 1–21—these papers shall be referred to in following as Part I and Part II respectively.

2) Part I, Theorems 1, 2 and 3.

$$(1) \quad \begin{cases} A = Ae_{1,1} + Ae_{1,2} + \cdots + Ae_{1,f(1)} + Ae_{2,1} + \cdots + Ae_{k,f(k)} + \mathfrak{l}_0 \\ \quad = e_{1,1}A + e_{1,2}A + \cdots + e_{1,f(1)}A + e_{2,1}A + \cdots + e_{k,f(k)}A + \mathfrak{r}_0, \end{cases}$$

where $\mathfrak{l}_0$ and $\mathfrak{r}_0$ are the left and the right annihilator ideals of $\sum e_{\varkappa,i}$ and are contained in N. Now, we have introduced the following definitions[1], which in fact can be reduced to the above ones in case of algebras[2]:

Let A have a unit element. We call A quasi-Frobeniusean if there is a permutation $\big(\pi(1), \pi(2), \dots, \pi(k)\big)$ of $(1, 2, \dots, k)$ such that

i) $e_{\varkappa,1} A$ has a unique simple right subideal $\mathfrak{r}_{\varkappa,1}$ and $\mathfrak{r}_{\varkappa,1}$ is operator-isomorphic to $e_{\pi(\varkappa),1} A/e_{\pi(\varkappa),1} N$,

ii) $Ae_{\pi(\varkappa),1}$ has a unique simple left subideal $\mathfrak{l}_{\pi(\varkappa),1}$ and $\mathfrak{l}_{\pi(\varkappa),1}$ is operator-isomorphic to $Ae_{\varkappa,1}/Ne_{\varkappa,1}$.

If moreover

iii) $f(\varkappa) = f\big(\pi(\varkappa)\big)$

then we call A Frobeniusean.

And we proved, as in the case of algebras, that if A is quasi-Frobeniusean then α) holds for every left ideal $\mathfrak{l}$ and right ideal $\mathfrak{r}$, and conversely; and that if A is Frobeniusean then not only α) but

$$\beta') \qquad d_l(\mathfrak{l}) = d_r\big(A/r(\mathfrak{l})\big), \qquad d_r(\mathfrak{r}) = d_l\big(A/l(\mathfrak{r})\big)$$

is the case for every $\mathfrak{l}$ and $\mathfrak{r}$, and conversely; where d_l and d_r are defined as follows: In case $\mathfrak{m}$ is a simple left module of A, $d_l(\mathfrak{m})$ is the rank of $\mathfrak{m}$ with respect to the quasi-field of operator-automorphisms. If however $\mathfrak{m}$ is not simple and has a composition series $\mathfrak{m} = \mathfrak{m}_0 \supset \mathfrak{m}_1 \supset \cdots \supset \mathfrak{m}_s = 0$, then we put $d_l(\mathfrak{m}) = \sum_{i=1}^{s} d_l(\mathfrak{m}_{i-1}/\mathfrak{m}_i)$. $d_r(\mathfrak{n})$ is defined similarly for a right module $\mathfrak{n}$.

Further, the following lemma will be of use[3]:

Lemma. In the above definitions of quasi-Frobeniusean and Frobeniusean rings, we can drop the last condition in ii); it follows automatically from i) and the first half of ii).

Now we come to our problem:

Theorem 1. Let the lattices of completely reducible left ideals and completely reducible right ideals in A, a ring satisfying the minimum condition, be anti-isomorphic respectively to those of right and left ideals containing the radical. Then A is quasi-Frobeniusean. If moreover the anti-isomorphisms satisfy, respectively, $d_l(\mathfrak{l}) = d_r(A/\mathfrak{r})$ *and* $d_r(\mathfrak{r}) = d_l(A/\mathfrak{l})$, *where in either case* $\mathfrak{l}$ *and* $\mathfrak{r}$ *correspond to each other, then A is Frobeniusean.*

For the proof we consider first the maximal numbers of independent

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1) See Part II.

2) Part I, Lemma 2. The lemma connects indeed the representation theory to the structural theory.

3) See the second REMARK in Part II, § 4 and a CORRECTION at the end of the same paper. To establish this lemma, we employ the arguments used in the first half of our proof to Part II, Theorem 7.

4) See G. Birkhoff, Combinatorial relations in projective geometries, Ann. Math. **36** (1935), p. 743-8.

atoms in the above lattices, which are modular and complemented. Then it follows easily that every $Ae_{\varkappa,i}$ $[e_{\varkappa,i}A]$ has a unique simple left [right] subideal $\mathfrak{l}_{\varkappa,i}$ $[\mathfrak{r}_{\varkappa,i}]$ and that $\mathfrak{l}_0=\mathfrak{r}_0=0$ (See (1)). Next we consider the numbers of (irreducible) projective geometries[4] into which our lattices are decomposed. Then we find that $\mathfrak{l}_{\varkappa,i}\cong\mathfrak{l}_{\lambda,j}$ $[\mathfrak{r}_{\varkappa,i}\cong\mathfrak{r}_{\lambda,j}]$ if and only if $\varkappa=\lambda$. Our theorem now follows readily from the above lemma.

Corollary. If the left and the right ideal lattices Λ and P of A are anti-isomorphic to each other, then A is quasi-Frobeniusean. If moreover the anti-isomorphism is such that $d_l(\mathfrak{l})=d_r(A/\mathfrak{r})$, where $\mathfrak{l}$ corresponds to $\mathfrak{r}$, then A is Frobeniusean.

On combining with our previous results quoted above, we have

Theorem 2. If there is an anti-isomorphism whatsoever between the left and the right ideal lattices Λ and P of A, then indeed the annihilation relation causes a one. If furthermore the given correspondence satisfies $d_l(\mathfrak{l})=d_r(A/\mathfrak{r})$, where $\mathfrak{l}$ and $\mathfrak{r}$ correspond to each other, then so does the one obtained from the annihilation.

If we consider vector moduli over A, we see readily[1]

Theorem 3. Let Λ_m and P_m be, respectively, the lattices of A-submoduli in m-dimensional left and right vector moduli $L=Au_1+Au_2+\cdots+Au_m$ and $R=v_1A+v_2A+\cdots+v_mA$ over A. If Λ_m is anti-isomorphic to P_m, then the annihilation with respect to scalar product (of a left vector with a right vector) gives really such an anti-isomorphism. If the given anti-isomorphism is such that $d_l(\mathfrak{L})=d_r(R/\mathfrak{R})$, where $\mathfrak{L}\leqq L$ and $\mathfrak{R}\leqq R$ correspond to each other, then the same relation holds for the anti-isomorphism caused by the annihilation.

Remark. In case of an algebra A the above conditions $d_l(\mathfrak{l})=d_r(A/\mathfrak{r})$, etc., can be replaced, as one easily sees, by more natural ones: $(\mathfrak{l}:F)+(\mathfrak{r}:F)=(A:F)$, etc.

Further, in the above theorems the provision that the ring satisfies the minimum condition (whence the maximum condition as well) can not be omitted; Consider for instance a valuation ring of an exponential valuation whose domain of values consists of all the rational numbers, say.

In the above connection, it is perhaps of some interest to note the following fact: *Group algebras* (modular or non-modular) and Grassmann-Cartan's *algebras of outer forms* constitute useful models of Frobeniusean algebras; in fact, group algebras are *symmetric* in the sense of R. Brauer and C. Nesbitt[2]. But, these algebras possess a well known second 1-1 correspondence between left and right ideals, namely, a (direct) isomorphism between left and right ideal lattices. Combining this with our anti-isomorhism, we see that both the left and the right ideal lattices of these algebras are *self-dual*. However, the same is certainly not the case for general (not only Frobeniusean but) symmetric algebras. This is evident in case the underlying field is not algebrai-

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1) For vector moduli cf. M. Hall, A type of algebraic closure, Ann. Math. **40** (1939), p. 360–9, and Part II, § 6.

2) R. Brauer-C. Nesbitt, On the regular representations of algebras, Proc. Nat. Acad. Sci. U. S. A. **23** (1937), p. 236–40. See also Part I, § 9.

cally closed, but the example can be given also with respect to an algebraically closed underlying field. In fact, even a much weaker symmetry between the upper and the lower Loewy parts of directly indecomposable components in the regular representations fails to prevail.

A complete account of the above theorems, together with other related results and examples, will be given elsewhere, as a continuation of the previous papers *On Frobeniusean algebras*, I and II.