PAPERS COMMUNICATED

94. *The exceptional Values of Functions with the Set of Capacity Zero of Essential Singularities.*

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1. Let $w=f(z)$ be a one-valued analytic or meromorphic function defined in an open domain D with boundary E.

Recently, among many authors, Prof. Kunugi has obtained some very interesting results concerning the values taken by $f(z)$ in D infinitely many times near a point $z=z_0$ of E and the well defined open set $S_{z_0}^{(D)}-S_{z_0}^{(E)}$, provided that $S_{z_0}^{(D)}-S_{z_0}^{(E)}$ does not reduce to an empty set[1].

Now, let us consider the following important case.

Suppose that $w=f(z)$ is a one-valued analytic or meromorphic function defined in an open domain D except for a closed set $E(\subset D)$ of essential singularities, namely, defined in the neighbourhood of E.

If the set E is of Carathéodory's linear measure zero, then the important theorem of Besicovitch[2] shows that the set $S_{z_0}^{(D)}$ is identical with the whole Gaussian plane and that, if z_0 is not an isolated point of E, $S_{z_0}^{(E)}$ becomes also identical with the whole Gaussian plane, that is, $S_{z_0}^{(D)}-S_{z_0}^{(E)}$ reduces to an empty set.

In such cases, we can proceed more metrically instead of rather topologically, and in the paper published in this Proceedings entitled "The exceptional Values of Functions with the Set of linear Measure Zero of essential Singularities," we have already proved that, if the set E is of Carathéodory's linear measure zero, then near each point of E, $f(z)$ takes all finite values except perhaps for a set of $1+\eta$-dimensional measure zero, where η is an arbitrary positive number.

We do not know yet whether the set of exceptional values in the theorem just stated may be reduced to be of linear measure zero or not.

But, roughly speaking, if E is sufficiently small (with respect to some measure whose dimensional order is not higher than that of linear measure), then we may naturally expect that the set omitted by $f(z)$ should be much smaller.

On the one hand, there is indeed a theorem established by M. L. Cartwright which is as follows[3]:

If E is of capacity zero, then $f(z)$ takes all values, except perhaps

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1) Kinjiro Kunugui. Sur un problème de M. A. Beurling, Proc. **16** (1940).

2) A. S. Besicovitch. On sufficient conditions for a function to be analytic, etc., Proc. London Math Soc. (2), Vol. 32 (1931). See also our quoted paper.

3) M. L. Cartwright. On the Behaviour of an Analytic Function in the Neighbourhood of its Essential Singularities, Theorem II c. Math. Annalen. Vol. 112 (1936). Cartwright states her theorem at first in terms of logarithmic measure, but her theorem remains still valid in the form cited above.

a set which, for each positive number ε, *can be enclosed in a sequence of circles whose diameters* $d_1, d_2, \cdots$ *satisfy the condition* $\sum_i h(d_i) < \varepsilon$, *where* $h(t)$ *is any increasing, continuous function such that*

$$h(0)=0 \quad and \quad \int_0^k \frac{h(t)}{t}\,dt < \infty \text{ for some positive } k.$$

On the other hand, R. Nevanlinna has obtained another very interesting theorem which is as follows[1]:

Suppose that $f(z)$ *is a one-valued analytic function defined in the whole finite plane* π *except for a bounded set* E *of capacity zero.*

If the set of all the finite values omitted by $f(z)$ *for* $\pi - E$ *is of capacity positive, then the function reduces to a constant.*

In other words, *if* E *is the essential singularities of* $f(z)$, *which is non-constant, then* $f(z)$ *takes, in* $\pi - E$, *all the finite values except a est of capacity zero.*

This result is of Liouville's type and it is desirable to obtain a more precise theorem of which the type is similar to that of our cited theorem or of Cartwright's, since, in Nevanlinna's proof, the hypothesis that the function $f(z)$ is regular in the whole finite plane except for E is essential[2].

The methods used in our preceding paper cited above will lead us to the results hoped for with very simple proofs, based upon, as lemmas, two fundamental theorems on the theory of potentials, which will be stated without proofs.

2. The theorem now we are to prove is:

Theorem A. *If* $w=f(z)$ *is a one-valued analytic function non-constant, defined in any open domain* D *except for a bounded set* $E(\subset D)$ *of essential singularities of capacitg zero, then,* $f(z)$ *takes all the finite values except perhaps those contained in a set of capacity zero.*

Lemma 1[3]. *Let* $\psi(z)$ *be a one-valued harmonic function defined for an open domain* D *except perhaps for a bounded set* $E(\subset D)$ *of capacity zero.*

If $\psi(z)$ *is bounded in* $D-E$, *then it becomes harmonic also at every point of* E.

Let F be a bounded closed set lying in Gaussian plane ω. Then the open set $\omega - F$ is decomposed into one or more (at most enumerable infinity) of its components, each of which is an open connected domain whose frontier is contained in F.

There is one among them that contains the point at infinity. We shall call this the *exterior domain* of F. If F does not contain any continuum, then the exterior domain of F is identical with the set $\omega - F$.

The Green's function of the exterior domain of F with the pole at infinity is characterized as follows[4]:

1) R. Nevanlinna. Eindeutige Analytische Funktionen (1936), p. 135, Theorem 3.

2) R. Nevanlinna. Loc. cit. p. 136.

3) P. J. Myrberg. Über die Existenz der Greenschen Funktionen auf einer gegebenen Riemannschen Fläche. Acta Math. 61 (1933). R. Nevanlinna. Loc. cit. p. 132, Theorem 2.

4) P. J. Myrberg. Loc. cit.

The Green's function is the smallest of all the positive harmonic functions defined in the exterior domain of F which have a logarithmic pole at infinity.

Moreover, it is known that the Green's function has an inferior limit zero at some boundary points of the domain[1].

Concerning the existence of such a function, the following theorem is fundamental:

Lemma 2[2]. *If a bounded closed set F is of capacity positive, then there exists Green's function of the exterior domain of F.*

Proof of Theorem A. Let F be the set of all finite values omitted by $f(z)$ in $D-E$.

As is shown by Miss Cartwright[3], the set F can not contain any continuum, since the set of capacity zero is of Carathéodory's linear measure zero[4].

Suppose that F were of positive capacity. Then, there would exist a bounded and closed subset M of positive capacity in F.

Now it is not difficult to find two disjoint, closed and bounded subsets F_1 and F_2 of M whose capacities are also positive.

Let Green's functions of the exterior domains of F_1 and F_2, whose existence is assured by Lemma 2, be $g_1(w)$ and $g_2(w)$ respectively. As F_1 and F_2 do not contain any continuum, each one of them is contained in the exterior domain of the other.

Consider now the following function:

$$\varphi(w) = g_1(w) - g_2(w) .$$

Since $g_i(w)$ is a positive and non-constant harmonic function in the exterior domain of F_i, which is identical with the complementary set of F_i, with logarithmic pole at infinity and is also bounded in any bounded open set containing $F_i (i=1,2)$, $\varphi(w)$ is harmonic not only at every finite point outside $F_0 = F_1 + F_2$, but also at infinity, for the poles of g_1 and g_2 are cancelled out with each other, which shows that $\varphi(w)$ is a bounded harmonic function defined for all the points of $\omega - F_0$.

Moreover, since, on the one hand, $g_1(w)$ has an inferior limit zero at a point of F_1 at which $g_2(w)$ is positive, while on the other hand, $g_2(w)$ has an inferior limit zero at a point of F_2 at which $g_1(w)$ is positive, we find that $\varphi(w)$ can be either positive or negative, which shows $\varphi(w)$ is not a constant.

Now let us consider the following function:

$$\psi(z) = \varphi[f(z)] .$$

As $f(z)$ does not take any value from F_0, $\psi(z)$ is certainly a well defined harmonic function in $D-E$.

Then, we find this bounded and harmonic in D except perhaps for

1) P. J. Myrberg. Loc. cit.

2) P. J. Myrberg. Loc. cit. G. Szegö. Bemerkungen zu einer Arbeit von Herrn M. Fekete. Math. Zeits. 21 (1924).

3) M. L. Cartwright. The exceptional values of functions with a non-enumerable set of essential singularities, Quart. J. Math., Vol. 8 (1937).

4) R. Nevanlinna. Loc. cit. p. 145, Theorem 2.

a bounded closed subset E of capacity zero.

Hence, by Lemma 1, $\psi(z)$ is harmonic at every point of E, if properly defined on E.

Since $\varphi(w)$ is not a constant, we can choose two values w' and w'' of $\omega-F_0$ such that

$$(*) \qquad \varphi(w') \neq \varphi(w'') .$$

Denoting by z_0 any point of the set E whose Carathéodory's linear measure is known to be zero, the values taken by $f(z)$ near z_0 is every where dense in the plane ω by Besicovitch's theorem[1]. Hence there exist in $D-E$ two sequences of points $\{z_i'\}$ and $\{z_i''\}$ both tending to the point z_0 and satisfying

$$f(z_i') \rightarrow w' \quad \text{and} \quad f(z_i'') \rightarrow w'' \text{ as } i \rightarrow \infty .$$

Then, it would follow

$$\psi(z_0) = \lim_{i\to\infty} \psi(z_i') = \lim_{i\to\infty} \varphi[f(z_i')] = \varphi(w')$$

and also

$$\psi(z_0) = \lim_{i\to\infty} \psi(z_i'') = \varphi(w'') ,$$

which is impossible on account of $(*)$, and this completes the proof.

Unter the same assumption of Theorem A, denoting any point of E by z_0, there exists a sequence of open domains $\{D_i\}$ such that

1° $D \supset D_1 \supset D_2 \supset \cdots \ni z_0$,

2° the diameter of D_i tends to zero as $i \rightarrow \infty$,

3° each $E_i = D_i E$ is a closed set, namely, it is contained in D_i with its limiting points.

These may be deduced from the fact that, since E is of Carathéodeory's linear measure zero, the projections of E on the coordinates-axes are of Lebesgue's linear measure zero, so that there are two systems of lines which are parallel to the axes, everywhere dense in D, and not meeting with the closed set E.

For each i, let us denote by F_i the set of the finite values omitted by $f(z)$ in $D_i - E_i$.

Then, by 1° and 3°, it follows that $F_1 \subset F_2 \subset \cdots$.

Writing $\widetilde{F} = \sum_{i=1}^{\infty} F_i$, we find the capacity of $\widetilde{F}$ is 0, since capacity of each F_i is zero by Theorem A.

Now it is obvious by 2° that any finite value not belonging to $\widetilde{F}$ is assumed by $f(z)$ infinitely many times near the point z_0.

From the above discussion, we have the following precisement of Theorem A:

Theorem B. *If $w=f(z)$ is a one-valued, analytic function, non-constant, defined in any open domain D except for a bounded closed set $E(\subset D)$ of essential singularities of capacity zero, then, near each point of E, $f(z)$ takes all the finite values infinitely many times except perhaps those belonging to a set of capacity zero.*

1) A. S. Besicovitch. Loc. cit.

We remark here that even if the point z_0 does not belong to D but to the boundary of D, still Theorem B holds good, provided that z_0 is not an isolated point of ED.

3. The theorems established above contain not only the theorem of Nevanlinna, but also that of Cartwright, since any set of capacity zero is of h-measure zero which appears in her theorem[1].

Though in this paper we have restricted ourselves to regular functions, it many be needless to say that our methods will enable us to treat meromorphic functions in a similar way.

It is also easy to see that, introducing spherical potentials and spherical capacity[2], or considering linear transformations, as usual, the restriction of boundedness of the set E in our theorems will be immaterial.

We owe kind remarks to Dr. Prof. Tsuji and we wish here to express our warmest thanks to him.

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1) See the note 4) of p. 431.

2) O. Frostman. Potentiel d'équilibre et capacité des ensembles (1935). pp. 62-64.