12. *An Abstract Integral, VI.*

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The purpose of this paper is to give an integral, similar to that of H. Freudenthal[1]. Our integral is defined for the functions with domains in a general lattice and range in a metric commutative semi-group.

1. We begin by the definitions and notations[2]:

[1.1] A is a metric commutative semi-group with zero elements whose operation is denoted by addition. And the addition is a contraction, i. e., $\delta(u+w, v+w) \leqq \delta(u, v)$, where $\delta(u, v)$ is the distance between u and v.

[1.2] L is a lattice with zero element.

[1.3] $f(x)$ is a one-valued function in L to A such as $f(0)=0$.

[1.4] A denumerable set $\{a_i\}=z(a)$ in L is called *resolution* of a if

1° $a_i > 0$, 2° $a_i \frown a_j = 0$ if $i \neq j$, 3° $\bigvee a_i = a$,

4° $\{a_i\}$ generates a Boolean algebra $L(z(a))$.

[1.5] $Z(a)$ is the class of all resolutions $z(a)$ of a.

[1.6] $z(a) \leqq z'(a)$ if and only if $L(z(a)) \leqq L(z'(a))$, the latter inequality being set implication.

[1.7] $y(a)$ is a finite subset of $z(a)$.

[1.8] $Y(z(a))$ is a class of all $y(a)$ such that $y(a) \leqq z(a)$.

[1.9] If $Z(a)$ consists of only one trivial resolution $z(a)=\{a\}$, then a is called *trivially soluble.*

[1.10] $y(a) \leqq y'(a)$ if and only if $y'(a)$ includes $y(a)$ as set.

[1.11] $f(y(a)) = \sum_{a_i \in y(a)} f(a_i)$.

Under above definitions we have clearly,

(1.12) $Z(a)$ is a partially ordered system.

(1.13) $Y(z(a))$ is a Moore-Smith set.

[1.14] If $f(y(a))$ converges to $u \in A$ in the sense of Moore-Smith, then we denote $u=f(z(a))$.

2. Here we define an integral as follows:

[2.1] If $f(z(a))$ converges to a unique $v \in A$ in $Z(a)$ in the sense of G. Birkhoff[3], we denote

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1) H. Freudenthal, Proc. Ned. Akad. Wet. Amsterdam, **39** (1936).

2) [ ] indicates axiom and definition, ( ) theorem.

3) G. Birkhoff and L. Alaoglu, Ann. of Math., **41** (1940), 293-309.

$$v=I(f:a)=\int_a f(dx)$$

and say that $f(x)$ is *integrable at* a, and v is the *integral of* f at a.

By the definitions 2.1 and 1.9 it is evident that

(2.2) If a is trivially soluble, then $I(f:a)=f(a)$.

Furthermore, we have the additivity of the integral, that is

(2.3) *If* f *and* g *are integrable at* a, *then* $f+g$ *is also and*

$$I(f:a)+I(g:a)=I(f+g:a).$$

Proof: We put $h(x)=f(x)+g(x)$. By the integrability of f and g $f\big(z(a)\big)$ and $g\big(z(a)\big)$ exist, and

$$f\big(y(a)\big)\in K\Big(f\big(z(a)\big)\,;\,\varepsilon\Big),\quad g\big(y(a)\big)\in K\Big(g\big(z(a)\big)\,;\,\varepsilon\Big)$$

for all $y(a)\geqq y_0(a)$, where $K(w;\varepsilon)$ is the sphere in A with center w and radius ε. Since addition is contraction, $f\big(y(a)\big)+g\big(y(a)\big)\in K\Big(f\big(z(a)\big)+g\big(z(a)\big)\,;\,2\varepsilon\Big)$. Hence we have

$$f\big(z(a)\big)+g\big(z(a)\big)=h\big(z(a)\big).$$

Since f and g are integrable, $z(a)$ has a successor $z'(a)$ whose successors lie in $K\big(I(f:a);\varepsilon\big)$. By the definition $z'(a)$ has also a successor $z''(a)$ whose all successors lie in $K\big(I(g:a);\varepsilon\big)$. Thus for all $z(a)\geqq z''(a)$ we have

$$f\big(z(a)\big)+g\big(z(a)\big)=h\big(z(a)\big)\in K\big(I(f:a);\varepsilon\big)+K\big(I(f:a);\varepsilon\big)$$
$$\leqq K\big(I(f:a)+I(g:a);2\varepsilon\big),$$

which proves the theorem.

If we assume that

[2.4] A has B as its operator-domain and B satisfies 1.1.

[2.5] $\delta(\alpha u,\alpha v)\leqq\delta(0,\alpha)\cdot\delta(u,v)$.

Then we have

(2.6) *If* $f(x)$ *is integrable at* a, *then* $\alpha f(x)$ *is also and*

$$I(\alpha f:a)=\alpha I(f:a).$$

Incidentally, A satisfies 2.4 and 2.5, then we can replace 1.11 by

[2.7] $f\big(y(a)\big)=\sum f(a_i)m(a_i)$, $a_i\in y(a)$, $m(a_i)\in B$.

In this case 2.3 and 2.6 hold also. We omit the proof.

3. In this section we suppose that

[3.1] L is a continuous geometry[1)].

J. von Neumann has proved that

(3.2) A denumerable set $\{a_i\}$ with $\bigvee a_i=a$ is a resolution of a if and only if $\{a_i\}$ is independent.

1) J. von Neumann, "Continuous Geometry," Princeton 1936.

(3.3) Let $\{a_i\}$ be independent and $\bigvee a_i = a$. If $z(a_i) = \{a_{ij}\}$, then $\{a_{ij}; i, j = 1, 2, \dots\}$ is a resolution of a.

Next we prove that

(3.4) *If* $g(x) = I(f:x)$ *exists for all* x *then* $g(a \cup b) + g(a \cap b) = g(a) + g(b)$.

It suffices to show the case: $a \cap b = 0$[1]. Let $z(a) = \{a_i\}$, $z(b) = \{b_i\}$ and $\{c_i\}$ be the set sum of $\{a_i\}$ and $\{b_i\}$. Then $z(a \cup b) = \{c_i\}$ is a resolution of $a \cup b$ by 3.3. Hence $f\big(z(a \cup b)\big) = f\big(z(a)\big) + f\big(z(b)\big)$. Since $g(x)$ exists for $x = a \cup b$, and by 2.1 we get the theorem.

Furthermore we can prove the complete additivity, that is,

(3.5) *If* $g(x)$ *exists for all* x, *then* $\sum g(a_i) = g(\bigvee a_i)$, *where* $\{a_i\}$ *are independent.*

Proof is similar to that of 3.4, hence we omit it.

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1) G. Birkhoff, " Lattice Theory," New York 1940. Theorem 4.13, p. 72.