30. *On some Property of Regular Functions in $|z|<1$.*

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§ 1. We shall introduce some of the directional maximum modulus of a regular function in the circle $|z|<1$, and give some theorem on it.

Let $f(z)$ be a regular function in $|z|<1$ and $M_\theta(r, \varepsilon) = \underset{|z|=r,\ \theta-\varepsilon<\operatorname{Arg} z<\theta+\varepsilon}{\text{l. u. b.}} |f(z)|$, ε being a positive number, and

$$\overline{\lim_{r\to 1}} \frac{M_\theta(r, \varepsilon)}{\varphi(r)} = \overline{M}_\theta(1, \varepsilon)_\varphi$$

$$\underline{\lim_{r\to 1}} \frac{M_\theta(r, \varepsilon)}{\varphi(r)} = \underline{M}_\theta(1, \varepsilon)_\varphi$$

where $\varphi(r)$ is a monotonously increasing function for $r\to 1$.

Now $$\underset{0<\varepsilon<\delta}{\text{g. l. b.}}\ \overline{M}_\theta(1, \varepsilon)_\varphi = \overline{M}_\theta(1)_\varphi\ ^{1)}$$

$$\underset{0<\varepsilon<\delta}{\text{l. u. b.}}\ \underline{M}_\theta(1, \varepsilon)_\varphi = \underline{M}_\theta(1)_\varphi\,.$$

These measures are of some use for a regular function in $|z|<1$. In the following we shall consider the case $\varphi(r)\equiv 1$ and denote by $\overline{M}_\theta(1)$ and $\underline{M}_\theta(1)$ respectively.

§ 2. Let E_θ be a set of θ, which is everywhere dense in $(0, 2\pi)$ and if $f(z)$ converges (to limits, ∞ included) for all θ, belonging to E_θ when $z=re^{i\theta}\to 1$, θ being fixed, then we shall call $f(z)$ *has F-property.*

Let E_θ be a set of θ, which is everywhere dense in $(0, 2\pi)$ and if $\overline{M}_\theta(1)=\infty$ for all θ, belonging to E_θ, then we shall call $f(z)$ *has M-property.*

Theorem: Let $f(z)$ be regular in $|z|<1$ and have F- and M-properties, then the Riemann surface of the inverse function of $f(z)$ has no parts of boundary in the finite plane[2].

By to have parts of boundary[3], having α, β as the end-points, in the finite plane, we shall mean the following:

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1) l. u. b. = least upper bound.
g. l. b. = greatest lower bound.

2) A sort of modular functions has *F*- and *M*-properties. *M*-property is equivalent to the unboundness of $|f(z)|$ in any sector.

3) The boundary of the domain within the angle $<\alpha p\beta$ may be a line of singularity or a set of limit points of branch points. We suppose here α and β both lie in the finite plane.

Let α and β be two accessible singular points when we prolong some element of the function on the Riemann surface along two straight lines respectively from a point p, then we can not prolong the element of the function on the Riemann surface, in the angle $<\alpha p\beta$, in any manner outside a certain domain lying in the limited part of the plane[1].

§ 3. Proof of the theorem: If there were a part of boundary, α and β being the end-points, consider the images $\overline{p'\alpha'}$ and $\overline{p'\beta'}$ of $\overline{p\alpha}$ and $\overline{p\beta}$ by $z=f^{-1}(w)$ respectively.

The curves $\overline{p'\alpha'}$ and $\overline{p'\beta'}$ converges to two points α' and β' (α' and β' may coincicle) on $|z|=1$ respectively.

For, $\overline{p'\alpha'}$, for instance, can neither oscillate infinitely often within $|z|\leqq\delta<1$, nor approach oscillating infinitely often to some arc on $|z|=1$ by the F-property. If it were so, let $\overline{0a}$ and $\overline{0b}$ be two radius vectors intersecting infinitely often the curve $\overline{p'\alpha'}$, and on which $f(z)$ tends to ξ and η respectively.

Since $f(z)\rightarrow\alpha$ along $\overline{p'\alpha'}$, ξ and η are both equal to α. Thus $|f(z)|$ is limited in some vicinity of the arc $\widehat{ab}$ on $|z|=1$ and $f(z)$ must be a constant by Koebe's theorem[2].

Now the first case; α' and β' are different.

However we may prolong some element in a domain bounded by $\overline{p'\alpha'}$, $\overline{p'\beta'}$ and $\overline{\alpha'\beta'}$ we can not prolong the element outside the domain on the Riemann surface bounded by $\overline{p\alpha}$, $\overline{p\beta}$ and $\widehat{\alpha\beta}$. Thus in the angle $<\alpha'0\beta'$ we have $\overline{M}_\theta(1)<K$ in a sufficiently small vicinity of the arc $\widehat{\alpha''\beta''}$ lying on $\widehat{\alpha'\beta'}$.

Next the second case; α' and β' coincide.

In this case we can prolong some element up to ∞ in any direction θ, except the set of θ of zero measure.

This comes from the method given by Gross.

We normalise the Riemann surface in the following way.

By $w_1=\dfrac{1}{w-p}$ the part of the star-region in the angle $<\alpha p\beta$ is transformed into a domain $\overline{G}$ on the w_1-plane such as ∞ into 0 and p into ∞.

By $w_1=\dfrac{1}{f(z)-p}=g(z)$, $\overline{G}$ is mapped on a simply connected domain G of the z-plane, G lying in a domain bounded by $\overline{p'\alpha'}$ and $\overline{p'\beta'}$.

Let $G(r)$ be the part of G for which $|z-\alpha'|<r$ and $|z|<1$, and $G(r,\varepsilon)$ the part of G for which $\varepsilon<|z-\alpha'|<r$.

In $\overline{G}$ there corresponds $\overline{G}(r)$ to $G(r)$, whose areal measure $J\big(\overline{G}(r)\big)$ is given by

$$\lim_{\varepsilon\to0}\int_{G(r,\varepsilon)}|g'(z)|^2dzd\bar{z}=\lim_{\varepsilon\to0}\int_{G(r,\varepsilon)}|g'(z)|^2rdrd\varphi \qquad (1)$$

1) For the functions having only F-property the theorem is not true, and it seems to me so for the functions having only M-property.

2) Tsuji: Hukuso Hensû Kansuron. Page 170.

where $z-\alpha'=re^{i\varphi}$.

$\int_{G(r,\varepsilon)} |g'(z)|^2 r dr d\varphi$ being bounded and monotonously increasing for $\varepsilon \to 0$ the integral (1) exists.

$\int_{G(r)} |g'(z)|^2 r dr d\varphi$, for r such as $r\to 0$, corresponds to the remainder of an integral which exists, hence

$$\int_{G(r)} |g'(z)|^2 r dr d\varphi \to 0 \quad \text{for} \quad r\to 0 .$$

To the set $\gamma(\rho)$ of G, which belongs to $|z-\alpha'|=\rho$ there corresponds a set $\bar{\gamma}(\rho)$ of $\bar{G}$ whose linear measure is given, when it is finite, by the integral $\int_{\gamma(\rho)} |g'(z)|\rho d\varphi$.

Now
$$\left(\int_{G(r)} |g'(z)|\rho d\rho d\varphi\right)^2 \leqq \int_{G(r)} |g'(z)|^2 \rho d\rho d\varphi \cdot \int_{G(r)} \rho d\rho d\varphi$$
$$\leqq J\left(\bar{G}(r)\right)\pi r^2 . \tag{2}$$

If
$$\varliminf_{\rho\to 0} \int_{\gamma(\rho)} |g'(z)|\rho d\rho = g > 0 , \quad \text{so we would}$$
have
$$\int_{G(r)} |g'(z)|\rho d\rho d\varphi \geqq gr .$$

This contradicts (2), for, $J\left(\bar{G}(r)\right)\to 0$ for $r\to 0$.

Now let us return to the star-region over the w-plane. Let $\gamma(\varphi, R)$ be the part of the set which corresponds to $\bar{\gamma}(\rho)$, for which $|w-p|\leqq R$.

Evidently for fixed R

$$\varliminf_{\rho\to 0} J\left(\gamma(\rho, R)\right)=0 , \quad J \text{ being the linear measure of } \gamma(\rho, R).$$

Every radial ray of the star-region, which ends at a point $\tilde{w}$, $|\tilde{w}-p|\leqq R$, (which is a branch-point), must meet $\gamma(\rho, R)$ for every ρ.

A sufficiently small vicinity of p, belonging to the star-region, if we measure the set of the radial rays by the measure of a point-set, at which the unit circle about p is intersected by the set of radial rays, so the measure of the above mentioned set of radial rays is given by

$$M(R)\leqq mJ\left(\gamma(\rho, R)\right) \tag{3}$$

when ρ is so small that $\gamma(\rho, R)$ does not appear in some vicinity of p.

Here m is a constant depending only on the area of this vicinity of p.

This is evident, for, $\gamma(\rho, R)$ is a sequence of analytic curves so far as $r>0$, and the least value is given when they meet perpendiculary to the radial rays of the star-region.

From (3) we have $M(R)=0$.

Thus the theorem is proved.