PAPERS COMMUNICATED

27. Notes on Banach Space (I): Differentiation of Abstract Functions.

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The object of this paper is to generalize the key theorem due to Pettis[1], from which almost all theorems concerning differentiation of the Banach space-valued functions of bounded variation are derived.

1. We will begin by some known definitions.

Let $\mathfrak{X}$ be a Banach space[2].

[1.1][1] If (γ_n) is a sequence in $\bar{\mathfrak{X}}$ (conjugate space of $\mathfrak{X}$) and $\mathfrak{Y}$ is a subset of $\mathfrak{X}$, then (γ_n) is called to have property $N(\mathfrak{Y})$ provided that $\|\gamma_n\| \leqq 1$ $(n=1,2,\ldots)$ and $\|y\| = \limsup_n |\gamma_n(y)|$ for every $y \in \mathfrak{Y}$. In this case we write symbolically $(\gamma_n) \in N(\mathfrak{Y})$.

[1.2][1] If X_R is an interval function on a Euclidean cube R_0 to $\mathfrak{X}$, then $\big(\pi_n \equiv (R_n, 1, \ldots, R_n, k_n)\big)$ is called a X_R-maximal sequence provided that π_n is a partition of R_0 and $\sum_{i=1}^{k_n} \|X_{R_{n,i}}\|$ tends to the total variation of X_R on R_0 as $n \to \infty$.

[1.3][1] If $\pi_n = (R_{n,1}, \ldots, R_{n,k_n})$ $(n=1,2,\ldots)$ is a sequence of partitions of R_0, then the linear closure of the set $(X_{R_{n,i}}; i=1,2,\ldots,k_n; n=1,2,\ldots)$ is called the X_R-span of (π_n).

[1.4][3] If x_s is a point function on R_0 to $\mathfrak{X}$, then x_s is called to be restricted provided that 1° x_s is bounded, 2° there exists a positive number M such that for any $\varepsilon > 0$ there corresponds a sequence of disjoint measurable sets $(e_1, e_2, \ldots)$ with $|e_i| < \varepsilon$, $\sum_{i=1}^{\infty} |e_i| = |R_0|$ and $\|\sum (x_{\xi_{i_k}} - x_{\eta_{i_k}})\| < R$ for any subset (i_k) and ξ_{i_k} and η_{i_k} in e_{i_k}.

[1.5][3] x_s is called to be measurable if there exists a sequence of restricted functions which tends to x_s. Symbolically we write $x_s \in M$.

[1.6] If x_s is integrable in the Birkhoff sense, then we write $x_s \in L$.

Concerning differentiation, we define

[1.7][1] If $\mathfrak{Z}$ is a subset of $\bar{\mathfrak{X}}$, then X_R is said to be $\mathfrak{Z}$-pseudo-differentiable to x_s almost everywhere provided that $\zeta(X_R)$ is differentiable (in the ordinary sense) to $\zeta(x_s)$ almost everywhere for all ζ in $\mathfrak{Z}$.

[1.8] X_R is (strongly) differentiable to x_s almost everywhere if for almost all s in R_0 $X_R/|I|$ tends to x_s when I is a cube containing s and $|I|$ tends to zero.

1) Pettis, Duke Math. Journ., **5** (1938).

2) [ ] denotes definition and ( ) theorem.

3) Jeffery, Duke Math. Journ.. **9** (1941).

If x_s is a strong derivative, then x_s is essentially separably valued. We define derivatives which need not be separably valued and is near to the strong derivative.

2.

(2.1) Let $\mathfrak{Y}$ be the linear closure of the set $(x_s; s\in R_0)$. If there exists a sequence $(\gamma_m)\in N(\mathfrak{Y})$ such that $\gamma_m(x_s)$ is measurable (in the ordinary sense) for all m, then $x_s\in M$.

Proof. We can suppose that x_s is bounded. Otherwise we put $E_m^n=\left(s;\ |\gamma_m(x_s)|>n\right)$ and $x_s^n=0$ in $E^n=\sum_{m=1}^{\infty}E_m^n$ and $=x_s$ otherwise. Since $\|x_s^n\|\leqq\limsup_m|\gamma_m(x_s^n)\|\leqq n$, and E_m^n tends to zero monotonously as $n\to\infty$, x_s^n tends to x_s almost everywhere by the property $N(\mathfrak{Y})$.

If x_s is bounded, it is sufficient to prove that x_s is restricted. Let $(e_1, e_2, \dots)$ be any sequence of disjoint measurable sets such that $|e_i|<\varepsilon$ and $\sum_{i=1}^{\infty}|e_i|=|R_0|$, and ξ_i, η_i be points in e_i. $(\gamma_m)\in N(\mathfrak{Y})$ implies

$$\|\sum_k(x_{\xi_{i_k}}-x_{\eta_{i_k}})\|\leqq\limsup_m\left|\gamma_m\Big(\sum_k(x_{\xi_{i_k}}-x_{\eta_{i_k}})\Big)\right|.$$

Now

$$\gamma_m\Big(\sum_k(x_{\xi_{i_k}}-x_{\eta_{i_k}})\Big)=\sum_k\Big(\gamma_m(x_{\xi_{i_k}})-\gamma_m(x_{\eta_{i_k}})\Big)$$

is bounded, for $\gamma_m(x_s)\equiv f_m(s)$ is a measurable (in the ordinary sense) function of s and is uniformly bounded. Thus $\limsup\gamma_m(x_s)=\limsup f_m(s)$ is also. This proves that x_s is restricted.

(2.2) Let $\mathfrak{Z}=(\gamma_j)\in N(\mathfrak{Y})$ for $\mathfrak{Y}$ in (2.1). If x_s is almost everywhere the $\mathfrak{Z}$-pseudo-derivative of $X(R)$, then x_s is measurable. Furthermore if X_R is of bounded variation, $x_s\in L$.

Proof. $\gamma_j(x_s)$ is a derivative (in the ordinary sense), and then it is measurable. This implies x_s is measurable by (2.1). As Pettis has proved, $\int\|x_s\|\,ds$ exists, which implies $x_s\in L$.

(2.3) If X_R is additive and of bounded variation and X_R is (γ_j)-pseudo-differentiable to x_s with $(\gamma_j)\in N(\mathfrak{Y})$, then $x_s\in L$. Further let $Z_R=X_R-\int_R x_s\,ds$ and $\mathfrak{W}$ be the Z_R-span of a Z_R-maximal sequence (π_n). If we suppose that X_R is (δ_j)-pseudo-differentiable to x_s for some $(\delta_j)\in N(\mathfrak{W})$, then that X_R is differentiable to x_s almost everywhere is equivalent to $\int_R x_s\,ds$ is differentiable to x_s almost everywhere.

Proof is done by (2.2) and the theorem (2.3) in the Pettis paper. As a particular case we get

(2.4) If X_R is additive and of bounded variation, and there exists $(\gamma_j)\in N(\mathfrak{X})$ such that X_R is (γ_j)-pseudo-differentiable to x_s, then $x_s\in L$ and that X_R is differentiable to x_s almost everywhere is equivalent to $\int_R x_s\,ds$ is differentiable to x_s almost everywhere.

(2.5) If X_R is additive and of bounded variation, and X_R is $\bar{\mathfrak{X}}$-pseudo-differentiable to x_s, then the conclusion of (2.4) holds.

3. We will now turn to the differentiation of point functions on the interval (0, 1) to Banach space $\mathfrak{X}$.

[3.1] If the integral $\int_0^s \frac{x_s - x_t}{(s-t)^{\frac{3}{2}}} dt$ exists, we put

$$D^{\frac{1}{2}} x_s = \frac{1}{2\sqrt{\pi}} \int_0^s \frac{x_s - x_t}{(s-t)^{\frac{3}{2}}} dt$$

integral being in the Birkhoff sense.

This is the definition of the fractional differentiation due to Riemann-Liouville in the real case.

[3.2] If $D^{\frac{1}{2}}(D^{\frac{1}{2}} x_s)$ exists, then we call it the (*)-derivative of x_s and we say that x_s is (*)-differentiable.

This derivative need not be essentially separably valued.

(3.3) If x_s is (*)-differentiable almost everywhere, then x_s is pseudo-differentiable almost everywhere to the same value.

(3.5) If x_s is the indefinite integral, that is $x_s = \int_0^s y_t dt$ such that $\int_0^1 \| y_t \| dt$ exists, then x_s is (*)-differentiable at s such that $\int_0^s \| y_t \| \frac{dt}{s-t}$ exists.

If the (*)-derivative of singular function is equal to zero almost everywhere, then x_s is almost everywhere differentiable to y_s in the set $\int_0^s \| y_t \| \frac{dt}{s-t}$ exists. This assumption is not true in general, but if we modify the kernel a little this holds. In the next section we will give a more general and better definition.

4. We will now replace (0, 1) by $(-\infty, \infty)$ for convenience.

[4.1] x_s is called to be (#)-differentiable at t if the limit

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} K_n(t-s) x_s ds,$$

exists, where the kernel satisfies the following conditions:

1°. For any real integrable (in the Lebesgue sense) function $g(t)$ and its integral $G(t) = \int_0^t g(s) ds$, $\int_{-\infty}^{\infty} K_n(x-s) G(s) ds$ tends to $g(t)$ almost everywhere.

2°. For any real singular function $h(t)$ $\lim_{n\to\infty} \int_{-\infty}^{\infty} K_n(t-s) h(s) ds = 0$ almost everywhere.

Such kernel exists clearly. This derivative is not also separably valued in general.

From (2.3) we can prove that

(4.2) If y_s is of bounded variation and is (γ_j)-pseudo-differentiable to x_s almost everywhere with $(\gamma_j) \in N(\mathfrak{Y})$, then $x_s \in L$. Further if we put $z_s = y_s - \int^{s} x_t dt$ and $Z_R = z_b - z_a$ for $R = (a, b)$ and then $\mathfrak{W}$ the Z_R-span of a Z_R-maximal sequence (π_n) and if we suppose that y_s is (δ_j)-pseudo-differentiable to x_s for some $(\delta_j) \in N(\mathfrak{Y})$, then y_s is (#)-differentiable to x_s almost everywhere.

(4.3) If y_s is of bounded variation and there exists $(\gamma_j) \in N(\mathfrak{X})$ such that y_s is (γ_j)-pseudo-differentiable to x_s almost everywhere, then $x_s \in L$ and y_s is ($\#$)-differentiable to x_s almost everywhere.

As an application we define

[4.4] If $\{x_\alpha\}$ is a (denumerable or not) set in $\mathfrak{X}$, then $\{x_\alpha\}$ is called the base if

1°. There exists a $\{\zeta_\alpha\} \subset \overline{\mathfrak{X}}$ such that $\{x_\alpha\}$ and $\{\zeta_\alpha\}$ are the biorthogonal system.

2°. For any $x \in \mathfrak{X}$ there are $(\zeta_{\alpha_i}) \subset \{\zeta_\alpha\}$ and $(x_{\alpha_i}) \subset \{x_\alpha\}$ such that $x = \sum_{i=1}^{\infty} \zeta_{\alpha_i}(x) x_{\alpha_i}$ uniquely.

Set of almost periodic functions has such a base.

(4.5) Suppose that $\mathfrak{X}$ has the base such that

1°. $\|x\| = \limsup |\gamma(x)|$ where lim sup is taken for γ such as $\|\gamma\| \leqq 1$, γ belongs to the linear closure of $(\zeta_{\alpha_i};\ i=1, 2, \ldots)$.

2°. When $x_s = \sum_{i=1}^{\infty} \zeta_{\alpha(s)_i}(x_s) x_{\alpha(s)_i}$, there exists $\frac{d}{ds}\ \zeta_{\alpha(s)}(x) = \zeta_{\beta(s)}(x)$ almost everywhere and $\dot{x}_s = \sum \zeta_{\beta(s)_i}(x_s) x_{\beta(s)_i}$ almost everywhere. Then if y_s is of bounded variation, then there exists ($\#$)-derivative of y_s almost everywhere and is equal to $\sum_{i=1}^{\infty} \zeta_{\beta(s)_i}(x_s) x_{\beta(s)_i}$.

5. We will conclude the paper by the theorem:

(5.1) If X_R is the ($\mathfrak{X}$)-integral (in the Pettis sense) of x_s and X_R is of bounded variation, then x_s is integrable in the Birkhoff sense and X_R is the Birkhoff integral of x_s.