49. On Krull's Conjecture Concerning Completely Integrally Closed Integrity Domains, II.

By Tadasi NAKAYAMA.
Department of Mathematics, Nagoya Imperial University.

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The case of partially ordered abelian groups being settled in Part I[1], let us turn to integrity domains; we want to obtain an integrity domain which is completely integrally closed but can never be expressed as an intersection of special valuation rings[2]. Our following construction depends however on that of Part I.

Let A be a complete Boolean algebra satisfying the condition in Part I, Lemma 1; there be a countable set of non-atomic non-zero elements v_i in A so that for any $a>0$ in A we have $a\geqq v_i$ for a suitable i[3]. Denote its representation space by $\Omega=\Omega(A)$. Then the lattice-ordered abelian group L_Ω of continuous functions on Ω, taking (rational) integers and $\pm\infty$ as values and finite except on nowhere dense sets, cannot, as was shown in Part I, be represented faithfully by (finite) real-valued functions (over any space). Now, let K be a field, and consider, abstractly, variables $x(\mathfrak{p})$ which are in one-one correspondence with the points $\mathfrak{p}$ in Ω. When $\{\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_s\}$ is a finite set of (distinct) points of Ω, a polynomial of the variables $x(\mathfrak{p}_1)$, $x(\mathfrak{p}_2)$, $\dots$, $x(\mathfrak{p}_s)$ over K will be called in the following a $\mathfrak{p}_1\mathfrak{p}_2\cdots\mathfrak{p}_s$-polynomial. Let $\{\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_t\}$ be a subsystem of $\{\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_s\}$. A $\mathfrak{p}_1\mathfrak{p}_2\cdots\mathfrak{p}_s$-polynomial $F(\mathfrak{p}_1\cdots\mathfrak{p}_s)\ \Big(=F\big(x(\mathfrak{p}_1), \dots, x(\mathfrak{p}_s)\big)\Big)$ is said to be reduced to a $\mathfrak{p}_1\cdots\mathfrak{p}_t$-polynomial $F(\mathfrak{p}_1\cdots\mathfrak{p}_t)$, when it becomes the latter by putting $x(\mathfrak{p}_{t+1})=\cdots=x(\mathfrak{p}_s)=1$; in symbol $F(\mathfrak{p}_1\cdots\mathfrak{p}_s)\rightarrow F(\mathfrak{p}_1\cdots\mathfrak{p}_t)$. Further, let P be a set of first category in Ω and suppose that for each finite system $\{\mathfrak{p}_1, \dots, \mathfrak{p}_s\}$ of points in Ω not belonging to P there is given a $\mathfrak{p}_1\cdots\mathfrak{p}_s$-polynomial $F(\mathfrak{p}_1\cdots\mathfrak{p}_s)$. If here $F(\mathfrak{p}\cdots\mathfrak{p}_s)\rightarrow F(\mathfrak{p}_1\cdots\mathfrak{p}_t)$ whenever $\{\mathfrak{p}_1, \dots, \mathfrak{p}_s\}\supset\{\mathfrak{p}_1, \dots, \mathfrak{p}_t\}$, we call this whole scheme a *polynomial series* on Ω and denote it by $\{F; P\}=\{F(\mathfrak{p}\cdots\mathfrak{p}); P\}$. Two polynomial series $\{F; P\}$ and $\{F'; P'\}$, such that $F(\mathfrak{p}_1\cdots\mathfrak{p}_s)=F'(\mathfrak{p}_1\cdots\mathfrak{p}_s)$ for every $\{\mathfrak{p}_1, \dots, \mathfrak{p}_s\}\subset\Omega-Q$, where Q is a set of first category containing P, P', will be called equivalent; we consider equivalent polynomial series as one and the same. The sum (product) of two polynomial series $\{F_1; P_1\}$ and $\{F_2; P_2\}$ is defined by taking $F_1(\mathfrak{p}_1\cdots\mathfrak{p}_s)+F_2(\mathfrak{p}_1\cdots\mathfrak{p}_s)\ \big(F_1(\mathfrak{p}_1\cdots\mathfrak{p}_s)F_2(\mathfrak{p}_1\cdots\mathfrak{p}_s)\big)$ for $\{\mathfrak{p}_1, \dots, \mathfrak{p}_s\}\subset\Omega-(P_1\smile P_2)$. Then the totality of polynomial series (the totality of classes of equivalent polynomial series, to be exact) forms a ring R_Ω,

1) T. Nakayama, On Krull's conjecture concerning completely integrally closed integrity domains, I., Proc. **18** (1942), 185.

2) See the papers cited in Part I. Cf. also Enzyklopädie der Math. Wiss. I_1, 11, p. 40.

3) For instance, let A be the complete Boolean algebra of regular open sets of the interval (0, 1).

which is obviously an integrity domain.

Lemma 1. R_Ω is completely integrally closed.

Proof. Let $\{F; P_1\}$, $\{G; P_2\}$ and $\{H; P_3\}$ be three non-zero polynomial series such that for every $\nu=1, 2, \dots$ $\{F; P_1\}^\nu \{H; P_3\}$ is divisible by $\{G; P_2\}^\nu$;

$$\{F; P_1\}^\nu \{H; P_3\} = \{G; P_2\}^\nu \{K^{(\nu)}; P^{(\nu)}\}.$$

Then we want to show that $\{F; P_1\}$ is divisible by $\{G; P_2\}$.

For this purpose, let P be a set of first category containing P_1, P_2, P_3 and all the $P^{(\nu)}$, such that

$$F(\mathfrak{p}_1 \cdots \mathfrak{p}_s)^\nu H(\mathfrak{p}_1 \cdots \mathfrak{p}_s) = G(\mathfrak{p}_1 \cdots \mathfrak{p}_s)^\nu K^{(\nu)}(\mathfrak{p}_1 \cdots \mathfrak{p}_s)$$

for $\{\mathfrak{p}_1, \dots, \mathfrak{p}_s\} \subset \Omega - P$. Since $\{G; P_2\}$, $\{H; P_3\}$ are non-zero, there is a finite system $\{\bar{\mathfrak{p}}_1, \dots, \bar{\mathfrak{p}}_r\}$ of points in $\Omega - P$ such that $G(\bar{\mathfrak{p}}_1 \cdots \bar{\mathfrak{p}}_r) \neq 0$ and $H(\bar{\mathfrak{p}}_1 \cdots \bar{\mathfrak{p}}_r) \neq 0$. For those $\{\mathfrak{p}_1, \dots, \mathfrak{p}_s\}$ ($\subset \Omega - P$) containing $\{\bar{\mathfrak{p}}_1, \dots, \bar{\mathfrak{p}}_r\}$ certainly $H(\mathfrak{p}_1 \cdots \mathfrak{p}_s) \neq 0$. But then $F(\mathfrak{p}_1 \cdots \mathfrak{p}_s)$ must be divisible by $G(\mathfrak{p}_1 \cdots \mathfrak{p}_s)$, since the polynomial domain $K[x(\mathfrak{p}_1), \dots, x(\mathfrak{p}_s)]$ is, as is well known, completely integrally closed. Let thus

$$(*) \qquad F(\mathfrak{p}_1 \cdots \mathfrak{p}_s) = G(\mathfrak{p}_1 \cdots \mathfrak{p}_s) L(\mathfrak{p}_1 \cdots \mathfrak{p}_s)$$

($\{\mathfrak{p}_1, \dots, \mathfrak{p}_s\} \supseteqq \{\bar{\mathfrak{p}}_1, \dots, \bar{\mathfrak{p}}_r\}$). Here $L(\mathfrak{p}_1 \cdots \mathfrak{p}_s)$ is uniquely determined, because $G(\mathfrak{p} \dots \mathfrak{p}_s) \neq 0$ too. Further, if $\{\mathfrak{p}_1, \dots, \mathfrak{p}_t\}$ is another set containing $\{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$ and if $\{\mathfrak{p}_1, \dots, \mathfrak{p}_s\} \supset \{\mathfrak{p}_1, \dots, \mathfrak{p}_t\}$, then the same relation as $(*)$ holds for this, and therefore, $L(\mathfrak{p}_1 \cdots \mathfrak{p}_s) \to L(\mathfrak{p}_1 \cdots \mathfrak{p}_t)$. As for those $\{\bar{\mathfrak{p}}_1, \dots, \bar{\mathfrak{p}}_s\}$ ($\subset \Omega - P$) not containing $\{\bar{\mathfrak{p}}_1, \dots, \bar{\mathfrak{p}}_r\}$, we define $L(\mathfrak{p}_1 \cdots \mathfrak{p}_s)$ by $L(\{\mathfrak{p}_1, \dots, \mathfrak{p}_s\} \cup \{\bar{\mathfrak{p}}_1, \dots, \bar{\mathfrak{p}}_r\}) \to L(\mathfrak{p}_1 \cdots \mathfrak{p}_s)$. Then $L(\mathfrak{p}_1 \cdots \mathfrak{p}_s)$ ($\{\mathfrak{p}_1, \dots, \mathfrak{p}_s\} \subset \Omega - P$) form a polynomial series $\{L; P\}$, as can readily be seen. Moreover, the relation $(*)$ holds for every $\{\mathfrak{p}_1, \dots, \mathfrak{p}_s\} \subset \Omega - P$. Thus

$$\{F; P_1\} = \{G; P_2\} \{L; P\}.$$

This proves that R_Ω is completely integrally closed.

Lemma 2. R_Ω is not an intersection of special valuation rings in its quotient field.

Proof. Consider the lattice-ordered abelian group L_Ω, additively written, of continuous functions on Ω taking integers and $\pm\infty$ as values and finite except on nowhere dense sets, alluded to above. With every $f \geqq 0$ in L_Ω we associate a polynomial series $\{x^f\} = \{x^f; N\}$ as follows: N is the nowhere dense set where f becomes $+\infty$, and for $\mathfrak{p}_1, \dots, \mathfrak{p}_s \in \Omega - N$

$$x^f(\mathfrak{p}_1 \cdots \mathfrak{p}_s) = x(\mathfrak{p}_1)^{f(\mathfrak{p}_1)} \cdots x(\mathfrak{p}_s)^{f(\mathfrak{p}_s)}.$$

Then evidently $\{x^{f_1}\}\{x^{f_2}\} = \{x^{f_1+f_2}\}$. Hence the semi-group of positive elements in L_Ω is embedded isomorphically into the multiplicative semi-group of R_Ω. Further, $f_1 \geqq f_2$ ($\geqq 0$) if and only if $\{x^{f_1}\}$ is divisible by $\{x^{f_2}\}$ in R_Ω. If R_Ω were an intersection of special valuation rings, then L_Ω would be represented faithfully by real valued functions contrary to our former result. Hence the lemma is proved.

We have therefore

Theorem 1. The integrity domain R_Ω of polynomial series over $\Omega=\Omega(A)$, A being a complete Boolean algebra satisfying the condition of Part I, Lemma 1 (See above), is completely integrally closed, but can never be expressed as an intersection of special valuation rings in its quotient field.

In connection with above, let us next solve another problem, though small, of Krull. Namely, in his paper in Math. Zeitschr. cited before Krull reserved decision whether every principal ideal in a completely integrally closed integrity domain is always an intersection of highest-dimensional primary ideals or not[1)2)]. We shall show that the answer is again negative.

Let A, $\Omega=\Omega(A)$, L_Ω and R_Ω be as above. We then consider a subring R_0 of R_Ω consisting of all those polynomial series $\{F;P\}$ satisfying the condition:

(**) if $\mathfrak{p}_1, \dots, \mathfrak{p}_s \notin P$ then there exist suitable neighborhoods $U_1 \dots,$ U_s of $\mathfrak{p}, \dots, \mathfrak{p}_s$, respectively, such that for $\mathfrak{q}_i \in U_i$, $\mathfrak{q}_i \notin P$ $(i=1, 2, \dots, s)$ the polynomial $F(\mathfrak{q}_1 \cdots \mathfrak{q}_s)$ is obtained from $F(\mathfrak{p}_1 \cdots \mathfrak{p}_s)$ simply by replacing $x(\mathfrak{p}_1), \dots, x(\mathfrak{p}_s)$ by $x(\mathfrak{q}_1), \dots, x(\mathfrak{q}_s)$.

That R_0 is really a subring of R_Ω is obvious. Moreover, if $\{F_1; P_1\}$, $\{F_2; P_2\}$ are two elements of R_0 and if the former, say, is divisible by the latter in R_Ω, then the same is the case in R_0 too. For, if $\{F_1; P_1\}=\{F_2; P_2\}\{F_3; P_3\}$ then $\{F_3; P_3\} \in R_0$; this is evident from the uniqueness of division in polynomial domains. From this remark follows that R_0 is, simultaneously with R_Ω, a completely integrally closed integrity domain.

Let now $\{F; P\}$ be a non-zero element in R_0, and $\mathfrak{p}$ be a point of Ω not belonging to P. For a finite system $\{\mathfrak{p}, \mathfrak{p}_1, \dots, \mathfrak{p}_s\}$ $(\mathfrak{p}_1, \dots, \mathfrak{p}_s \notin P)$, containing $\mathfrak{p}$, we consider the highest power of $x(\mathfrak{p})$ which divides the polynomial $F(\mathfrak{p}\mathfrak{p}_1 \cdots \mathfrak{p}_s)$. Denote its exponent by $f_\mathfrak{p}(\mathfrak{p}\mathfrak{p}_1 \cdots \mathfrak{p}_s)$. If $\{\mathfrak{p}, \mathfrak{p}_1, \dots, \mathfrak{p}_s\} \subset \{\mathfrak{p}, \mathfrak{p}_1, \dots, \mathfrak{p}_t\}$ $(\mathfrak{p}_1, \dots, \mathfrak{p}_t \in P)$ then $f_\mathfrak{p}(\mathfrak{p}\mathfrak{p}_1 \cdots \mathfrak{p}_s) \geqq f_\mathfrak{p}(\mathfrak{p}\mathfrak{p}_1 \cdots \mathfrak{p}_t)$. Hence there is a finite system $\{\mathfrak{p}, \mathfrak{p}_1, \dots, \mathfrak{p}_s\}$ such that for any $\{\mathfrak{p}, \mathfrak{p}_1, \dots, \mathfrak{p}_t\}$ containing it $f_\mathfrak{p}(\mathfrak{p}\mathfrak{p}_1 \cdots \mathfrak{p}_t)=f_\mathfrak{p}(\mathfrak{p}\mathfrak{p} \cdots \mathfrak{p}_t)$. We denote this value by $f_\mathfrak{p}$. Thus with every point $\mathfrak{p} \in \Omega - P$ we have associated a non-negative integer $f_\mathfrak{p}$.

From the condition (**), which our $\{F; P\}$ sasisfies, follows that $f_\mathfrak{p}$ is continuous in $\Omega - P$. Therefore[3)], there exists a continuous function $f(\mathfrak{p})$ on the whole Ω taking integers and $\pm\infty$ as values, such that $f(\mathfrak{p})=f_\mathfrak{p}$ for $\mathfrak{p} \in \Omega - P$. $f(\mathfrak{p})$ is finite except on a nowhere dense set, and is an element of the lattice-ordered group L_Ω.

So, to every element $\{F; P\}$ in R_0 there corresponds an element $f=f(\mathfrak{p})$ of L_Ω. If $\{F; P\}$, $\{G; P'\} \in R_0$, $\{H; P''\}=\{F; P\}+\{G; P'\}$ and if they correspond respectively to f, g and h in L_Ω, then $h \geqq f \frown g$, because $h_\mathfrak{p} \geqq \mathrm{Min}(f_\mathfrak{p}, g_\mathfrak{p})$ for every $\mathfrak{p} \notin P \smile P'$. Further, the product

1) W. Krull, Beiträge zur Arithmetik kommutativer Integritätsbereiche, Math. Zeitschr. **41** (1936) (p. 669, Footnote 9).

2) A primary ideal is called highest-dimensional when it belongs to a minimal prime ideal.

3) Observe that Ω is the representation space of a complete Boolean algebra. Cf. T. Ogasawara, l. c.

$\{F; P\}\ \{G; P'\}$ corresponds to the sum $f+g$. Hence, if $\mathfrak{p} \in \Omega$ the totality $\mathfrak{a}_{\mathfrak{p}}$ of those elements $\{F; P\}$ in R_0 such as $f(\mathfrak{p}) = +\infty$ is an ideal of R_0, and indeed a prime ideal. $\mathfrak{a}_{\mathfrak{p}}$ is further not void.

Now, consider a minimal prime ideal $\mathfrak{P}$ in R_0. We make distinction between two possibilities[1]: i) $\mathfrak{a}_{\mathfrak{p}} \subseteqq \mathfrak{P}$ for a certain $\mathfrak{p} \in \Omega$; ii) there is no such $\mathfrak{p}$. In the first case we have, since $\mathfrak{P}$ is minimal, $\mathfrak{a}_{\mathfrak{p}} = \mathfrak{P}$. In the second case, there exists for every $\mathfrak{p} \in \Omega$ an element $\{F^{(\mathfrak{p})}; P^{(\mathfrak{p})}\}$ of $\mathfrak{a}_{\mathfrak{p}}$ not belonging to $\mathfrak{P}$. $f^{(\mathfrak{p})} \in L_\Omega$ corresponding to $\{F^{(\mathfrak{p})}; P^{(\mathfrak{p})}\}$ takes the value $+\infty$ at $\mathfrak{p}$, and there is a neighborhood $U_{\mathfrak{p}}$ of $\mathfrak{p}$ so that $f^{(\mathfrak{p})}(\mathfrak{q}) > 1$ for $\mathfrak{q} \in U_{\mathfrak{p}}$. Ω is covered by a finite number of such neighborhoods: $\Omega = U_{\mathfrak{p}_1} \cup U_{\mathfrak{p}_2} \cup \cdots \cup U_{\mathfrak{p}_n}$. Consider the product

$$\{F; P\} = \{F^{(\mathfrak{p}_1)}; P^{(\mathfrak{p}_1)}\}\ \{F^{(\mathfrak{p}_2)}; P^{(\mathfrak{p}_2)}\} \cdots \{F^{(\mathfrak{p}_n)}; P^{(\mathfrak{p}_n)}\}\ .$$

This does not belong to $\mathfrak{P}$, since $\mathfrak{P}$ is prime. But its corresponding element $f = f^{(\mathfrak{p}_1)} + f^{(\mathfrak{p}_2)} + \cdots + f^{(\mathfrak{p}_n)}$ of L_Ω is > 1 everywhere in Ω.

Let $\{X\} = \{X; 0\}$ be the polynomial series such as $X(\mathfrak{p}_1 \cdots \mathfrak{p}_s) = x(\mathfrak{p}_1) \cdots x(\mathfrak{p}_s)$ for every $\{\mathfrak{p}_1, \cdots, \mathfrak{p}_s\}$; 0 being the void set. When 1 is the function on Ω identically equal to 1 then this is nothing but $\{x^1\}$ in our former notation, and indeed, 1 is the element of L_Ω which corresponds to $\{X\} \in R_0$ in our sense. In the above case i) evidently $\{X\} \notin \mathfrak{P}$. But $\{X\} \notin \mathfrak{P}$ also in the second case. For, $\{F; P\}$ is divisible by $\{X\}$ and $\{F; P\} \notin \mathfrak{P}$ (see above).

Thus always $\{X\} \notin \mathfrak{P}$. This is the case for every minimal prime ideal $\mathfrak{P}$ in R_0, whence $\{X\}$ is contained in no highest-dimensional primary ideal in R_0. But $\{X\}$ is certainly not a unit in R_0. So we arrive at

Theorem 2. Let A be as before. Let R_0 be the integrity domain consisting of all the polynomial series satisfying the condition $(**)$, and $\{X\}$ be the element of R_0 such that $X(\mathfrak{p}_1 \cdots \mathfrak{p}_s) = x(\mathfrak{p}_1) \cdots x(\mathfrak{p}_s)$ for every $\{\mathfrak{p}, \cdots, \mathfrak{p}_s\}$. Then R_0 is completely integrally closed, but the principal ideal $(\{X\})$ is not an intersection of highest-dimensional primarg ideals.

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1) As a matter of fact, this first possibility is excluded. See Part I, Remark 1.