88. On the Cauchy's Integral Theorem.

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(Comm. by S. KAKEYA, M.I.A., Oct. 12, 1942.)

S. Pollard have obtained a following theorem, for extension of the Cauchy's well-known theorem[1]:

Theorem. *Let C be any closed plane jordan curve with no multiple points, and let D be a connected domain enclosed by C in its interior. Let further $f(z)$ be a uniform function defined in D and satisfy the following conditions:*

1° *The real and imaginary parts of $f(z)$ have partial derivative which satisfy Cauchy's equations at all points within D, and are integrable over every rectangle within D... integrability being understood either in the sense of Riemann, or more general sense of Lebesgue.*

2° *$f(z)$ is continuous on C so far as values at points within and on it are concerned.*

3° *C is a curve of bounded variation.*

Then the integral of $f(z)$ round the contour C is zero, that is

$$\int_C f(z)\,dz=0$$

But the proof of this theorem given by S. Pollard[1] seems to us to be insufficient for the general case[2]. The object of this paper is to give a correct proof of this theorem which modifies and simplifies Pollard's proof.

First, let us give certain lemmas.

Lemma 1. *Suppose that C be a rectifiable plane curve with no multiple point and denote its length by L. Then, for any positive number ε, there exists a polygon π inside C, which satisfies the following conditions:*

(1°) *Its sides are parallel to one or other of the axes.*

(2°) *It is possible to divide C and π, into equal number n of small arcs $C_1, C_2, \dots, C_n$ and broken lines $\pi_1, \pi_2, \dots, \pi_n$ respectively, so that, for each pair (C_i, π_i) $(i=1, 2, \dots, n)$, hold the inequality*

$$\rho(a, b) < \varepsilon \quad as \quad a \in C_i,\ b \in \pi_i$$

and that $n\varepsilon < 4L$.

(3°) *Denoting by $l(\pi)$ the length of π, we have $l(\pi) < 11L$: $l(\pi)$ is therefore uniformly bounded.*

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1) S. Pollard: On the conditions for Cauchy's theorem, proceedings of the London Math. Soc. Second Series, vol. **21** (1923), p. 456–482. Cf. also, E. Kamke: Zu dem Integralsatz von Cauchy, Math. Zeitschrift, Bd. **35** (1932), p. 535–543; J. L. Walsh: Approximation by polynomials in the complex domain, Paris, 1935, p. 9.

2) For example, consider the case where C has an angular point with angle wich is sufficiently small, and one of the tangents at this point is parallel to one or other of the axes. In this case, their non-consecutive links surely overlap, and the chain is not "regular."

Demonstration. Let us take any one interior point P of D and fix it. Associated with any positive number ε, take a number n such as $2L/n > \varepsilon/2 > L/n$ and take any initial point P_1 on C and obtain the points $P_2, P_3, \dots, P_n$ whose distance from P_1 measured along the curve C are respectively $L/n, 2L/n, \dots, (n-1)L/n$.

Denote the arcs $P_1P_2, P_2P_3, \dots, P_nP_1$ by $C_1, C_2, \dots, C_n$ respectively and let δ be the minimum distance between non consecutive C_i's. Join P and P_i by a jordan simple curve Γ_i in D.

Corresponding to each point P_i, construct a square S_i with center at P_i, whose sides are parallel to one or other of the axes, disjoint from any $\Gamma_j (j \neq i)$, and of side length less than $\delta/\sqrt{2}$.

Denote by C'_i the part of C_i whose one end point is a first point of intersection C_i and S_i and other end point is the last point of intersection of C_i and S_{i+1}, considered along C_i with its initial point P_i. Let the minimum distance between the arcs C'_i each other and between C'_i and $\sum_{j=1}^{n} \Gamma_j$ be denoted by δ'.

Devide each C'_i into at least two small arcs with length less than $\delta'/2$. And, corresponding to each of these small arcs, construct a square with its sides parallel to one or other of the axes, the center of these squares being at the center of the arcs. Furthermore, we suppose that the side length of these squares is greater than the length of small arc and less than the minimum of the double of this length and $\delta'/\sqrt{2}$.

Remark that each square attached to small arcs on C'_i is disioint from each of $\Gamma_j \ (j=1, 2, 3, \dots, n)$, and since the diagonals of all these squares are of length less than $\delta'/2$, no square attached to C'_i can touch a square attached to $C'_j (j \neq i)$.

The square attached to the small arcs of any given C_i overlap two by two and forms together a connected region Δ_i entirely enclosing C_i and with its boundary consisting of certain number of polygons, whose sides are parallel to one or other of the axes.

We obtain, from the above construction,

$$\delta' \leqq \delta < \varepsilon/2$$

Furthermore, we can say that the distance between any point on C'_i and any point on these boundary polygons is less than ε. In fact, let a be any point of C_i and b be any point on these polygons, and Q be the center of a square which contains the point b, then

$$\rho(a, Q) \leqq L/n < \ /2, \qquad \rho(b, Q) < \delta < \varepsilon/2$$
$$\rho(a, b) \leqq \rho(a, Q) + \rho(Q, b) < \varepsilon/2 + \varepsilon/2 = \varepsilon \tag{1}$$

Let us consider the whole region $\sum_{i=1}^{n} \Delta_i$. This total sum forms a connected region and its boundary forms a finite number of polygons whose sides are composed of segments parallel to one or other of the axes. Among these boundary polygons, there exists only one polygon which contains the fixed point P in its interior and is situated inside C. Denote this polygon by π, then we can conclude that the polygon π is just what we have demanded

In fact, the polygon π contains surely a part of contour of Δ_i, for, if we denote by Q_i the last point of the intersection of Γ_i and S_i, considered along the curve Γ_i from the point P_i, then no point on the arc Q_iP of the curve Γ_i can touch to any square. Thus, this arc is contained inside π and Q_i is a point on the polygon π and Δ_i.

From the construction of the small squares on the arcs C_i and C'_{i-1}, we can see that a part of the square S_i in the vicinity of Q_i on the boundary of Δ_i is situated outside all small squares attached to C'_i and C'_{i-1}. Thus, this part is contained on the polygon π.

Denote the part of π between Q_i and Q_{i+1} on the boundary of Δ_i, by π_i. Finally, we obtain n arcs $C_1, C_2, \dots, C_n$ on C and n broken lines $\pi_1, \pi_2, \dots, \pi_n$ on π. Each pair (C_i, π_i) $(i=1, 2, \dots, n)$ satisfies the condition (2°) for the sake of the inequality (1).

And next, to demonstrate the condition (3°), let us estimate the length of π. The side length of small squares attached to the arc C'_i is less than the double of the length of the small arc, and thus, the sum of contours of all squares attached to C'_i is less than eight time of length of C'_i. The length of contour of end square is less than $2\sqrt{2}\,\delta$ or than $2\sqrt{2}\,L/n$.

Since the length of π is less than the sum of length of the contours of all squares considered, we have thus

$$l(\pi) < 8\sum l(C_i) + n\cdot 2\sqrt{2}\,L/n < 8L + 3L = 11L \qquad \text{Q. E. D.}$$

Lemma 2. *Suppose that $f(z)$ is uniform and continuous in a closed domain $\bar{D}$ which is bounded by a rectifiable plane curve C with no multiple point. Then, for any given positive number η, there exists a polygon π enclosed by C, with its sides parallel to one or other of the axes, and such that the inequality*

$$\left| \int_C f(z)dz - \int_\pi f(z)dz \right| < \eta$$

is hold.

Demonstration. Since the function $f(z)$ be continuous in a closed domain $\bar{D}$, it is uniformly continuous in $\bar{D}$. Thus we have, for any positive number λ, there exists a suitable positive number ε which depends only on λ, such that the inequality

$$|f(z') - f(z'')| < \lambda \tag{2}$$

is hold for any point pair z', z'' in $\bar{D}$ such that $|z'-z''| < \varepsilon$.

Let us apply the lemma 1 for the curve C and the positive number $\varepsilon/2$, then we can construct a polygon π, and satisfy the following condition:

We can divide the curve C, into n small arcs $C_1, C_2, \dots, C_n$, with its length less than $\varepsilon/2$ and n broken line $\pi_1, \pi_2, \dots, n_n$ by their cut points $Q_1, Q_2, \dots, Q_n$, so that, for any one pair (C_i, π_i) and any two points a, b where $a \in C_i, b \in \pi_i$, the inequality

$$\rho(a, b) < \frac{\varepsilon}{2}$$

hold.

Denote now, by γ_i the closed (but not necessarily simple) curve formed by C_i the line $P_{i+1}Q_{i+1}$, π_i (described in the sense opposite to C_i) and the line Q_iP_i.

Consider the integral along γ_i. Some of the line P_iQ_i may lie partly outside C, and so the behaviour of $f(z)$ on them is not altogether assured. In this case, we replace $f(z)$ by an integrand $\phi(z)$ obtained as follow.

$$\begin{aligned}\phi(z)&=f(z);\ \text{on } C_i \text{ and } \pi_i\\ &=l(z);\ \text{on } P_iQ_i \text{ and } P_{i+1}Q_{i+1}\end{aligned}$$

where $l(z)$ is a linear function which, for example on P_iQ_i, coincides with $f(z)$ at p_i and at Q_i. Then evidently we have

$$\begin{aligned}\sum_{i=1}^{n}\int_{\gamma_i}\phi(z)dz&=\int_C\phi(z)dz-\int_\pi\phi(z)dz\\ &=\int_C f(z)dz-\int_\pi f(z)dz\end{aligned}$$

the line P_iQ_i being described twice, once in each direction, and the integral along them destroying one another.

Now we have by lemma 1, for any point on γ_i, $\rho(P_i, z)<\varepsilon$, thus obtain by inequality (2)

$$|\phi(z)-f(P_i)|<\lambda$$

Put $\phi(z)=f(P_i)+[\phi(z)-f(P_i)]$, then we have

$$\begin{aligned}\int_{\gamma_i}\phi(z)dz&=\int_{\gamma_i}f(P_i)dz+\int_{\gamma_i}[\phi(z)-f(P_i)]dz\\ &=\int_{\gamma_i}[\phi(z)-f(P_i)]dz \quad \text{for} \quad \int_{\gamma_i}f(P_i)dz=0\end{aligned}$$

Hence

$$\begin{aligned}\left|\sum_{i=1}^{n}\int_{\gamma_i}\phi(z)dz\right|&\leqq\sum_{i=1}^{n}\int_{\gamma_i}|\phi(z)-f(P_i)|\,ds\\ &\leqq\lambda\sum_{i=1}^{n}l(\gamma_i)\end{aligned}$$

and by lemma 1

$$\begin{aligned}\sum_{i=1}^{n}l(\gamma_i)&<11L+L+2\sum P_iQ_i\\ &<11L+L+2n\varepsilon<11L+L+8L=20L\end{aligned}$$

thus finaly we have

$$\left|\int_C f(z)dz-\int_\pi f(z)dz\right|<20L\lambda$$

Since λ is arbitraly number, now put $\lambda=\dfrac{\eta}{20L}$ then we have the initial inequality. Q. E. D.

Using the S. Pollard' and D. Menchoff's[1] results, we have the following lemma.

Lemma 3. *Suppose that $f(z)$ be uniform and continuous in a closed rectangle R whose sides are parallel to one or other of the axes. And its real and imaginary parts give partial derivatives which satisfy Cauchy's equations at all point within the contour ... integrability being understood either on the sense of Riemann, or in the most general sense of Lebesgue.*

Then we have

$$\int_R f(z)dz=0$$

Now it is easy to demonstrate our theorem by means of the lemmas.

Demonstration of theorem. By hypothesis, $f(z)$ is uniform and continuous on and inside C. For any positive number η, there exists a polygon π inside C, such that the inequality

$$\left|\int_C f(z)dz-\int_\pi f(z)\,dz\right|<\eta \tag{3}$$

hold, by lemma 2.

Since the sides of the polygon are parallel to one or other of the axes, it is possible to divide the polygon into finite number of rectangles with its sides parallel to one or other of the axes. Thus

$$\int_\pi f(z)dz=\sum\int_R f(z)dz \tag{4}$$

and by lemma 3

$$\int_R f(z)dz=0 \tag{5}$$

Since η is arbitraly, and by (3), (4), (5) finaly we have

$$\int_C f(z)dz=0 \qquad \text{Q. E. D.}$$

Remark 1. The condition 1° in our theorem is used only for the demonstration of the equality (see lemma 3)

$$\int_R f(z)\,dz=0$$

where R is a rectangle with its sides parallel to one or other of the axes and is unnecessary for evaluation of the difference

$$\int_C f(z)dz-\int_\pi f(z)dz$$

which tend to zero as $\pi\to C$.

On the other hand, in the rectanglar case, by D. Menchoff's result, the condition 1° is replaced by the new condition.

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1) D. Menchoff: Les conditions de menogéneité, Paris Hermann & c^{ie}, Editeurs, 1936.

I′ *Save for a set of values of (superficial) measure zero, the partial derivative of the function* $f(z)=u+iv$, $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ *exist in D and these derivative are summable, and save for a set of values of measure zero in D these partial derivatives satisfy Cauchy's equations.*

And so, the same condition hold in general case.

Remark 2. Adding the continuity of $f(z)$, to the condition 1° of this theorem, this generalized theorem which give us the equality $\int_C f(z)dz=0$, allow us to conclude, by Morera's theorem, that $f(z)$ is holomorphic in D:

The continuous function $f(z)=u+iv$ *is holomorphic in D, if the partial derivatives* $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ *exist with their derivative are summable, their values are finite and satisfy the conditions*

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}, \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$

save for a set of values of measure zero.